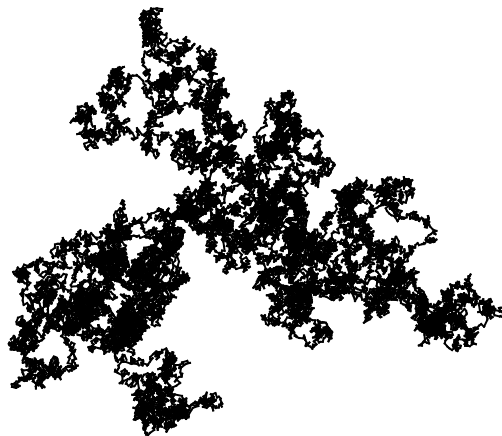


On Quadratic BSDEs with Final Condition in \mathbb{L}^2

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Chapter 1

Introduction

We are concerned with \mathbb{R} -valued backward stochastic differential equations (BSDEs) in the continuous semimartingale framework

$$Y_t = \xi + \int_t^T (F(s, Y_s, Z_s) dA_s + g_s d\langle N \rangle_s) - \int_t^T (Z_s dM_s + dN_s), \quad (1.1)$$

where the generator $F(t, y, z)$ has at most quadratic growth in z and g is a progressively measurable integrable process. For this reason, (1.1) is called *quadratic*. We call an adapted process (Y, Z, N) a *solution* to (1.1) if Y is continuous, Z is progressively measurable and integrable with respect to the fixed continuous local martingale M , and N is a continuous local martingale strongly orthogonal to M . In particular, if the filtration is generated by a Brownian motion W , (1.1) becomes the classic BSDE with $A_t = t$, $M = W$ and $N = 0$.

Let us recall that, quadratic BSDEs are first studied by Kobylanski [22]. Existence and uniqueness, comparison theorem and stability results are proved, when the terminal value is bounded. Later, Briand and Hu [8], [9] extend the existence result by assuming that the terminal value has exponential moments integrability. Moreover, a uniqueness result is obtained given a convexity condition as an additional requirement. Afterwards, Morlais [26] and Mocha and Westray [25] extend all these results to continuous semimartingale setting under rather strong assumptions on the generator. Recently, for Brownian framework, Bahlali et al [1] constructs a solution to quadratic BSDEs with the terminal value in \mathbb{L}^2 and the generator $F(t, y, z)$ satisfying \mathbb{P} -a.s. for all $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$,

$$|F(t, y, z)| \leq \alpha + \beta|y| + \gamma|z| + f(|y|)|z|^2, \quad (1.2)$$

for some $\alpha, \beta, \gamma \geq 0$ and $f(|\cdot|) : \mathbb{R} \rightarrow \mathbb{R}$ which is integrable and bounded on compact subsets of \mathbb{R} . However, as to the uniqueness of a solution, only purely quadratic BSDEs are studied.

As a natural extension of these works, this paper is devoted to answering the following questions: **1.** Does existence and uniqueness hold for BSDEs satisfying (1.2) with terminal value in \mathbb{L}^p for a certain $p > 0$? **2.** Can one establish the solvability of quadratic semimartingale BSDEs in a more general way under weaker assumptions ?

In **Chapter 2** we address the first question. We prove an existence result, by merely assuming that the generator is monotonic at $y = 0$ and has a linear-quadratic growth in z of type (1.2), and that the terminal value belongs to \mathbb{L}^p for a certain $p > 1$. To establish the a priori estimates, we use a combination of the estimates developed by Bahlali et al [1] and \mathbb{L}^p -type estimates developed by Briand et al [6]. Thanks to the estimates, we prove an existence result based on the localization procedure developed by Briand and Hu [8], [9]. The second contribution of this chapter is the uniqueness result. In the spirit of Da Lio and Ley [11] or Briand and Hu [3], we prove comparison theorem, uniqueness and a stability result via θ -technique under a convexity assumption. It turns out that our results of existence and uniqueness not simply provide wider perspectives on quadratic BSDEs but also, by setting $f(|\cdot|) = 0$, concern non-quadratic BSDEs studied in [27], [7], [6], [4], etc.

Chapter 2 is organized as follows. In Section 2.1, 2.2, we introduce basic notions and present auxiliary results. Section 2.3 proves the Itô-Krylov formula and a generalized Itô formula for $y \mapsto |y|^p (p \geq 1)$. The former one is used to treat discontinuous quadratic generators or discontinuous quadratic growth and the later one is used for a \mathbb{L}^p -type estimate. Section 2.4 reviews purely quadratic BSDEs and their natural extensions, based on Bahlali et al [4]. Section 2.5 studies existence, uniqueness and a stability result. Finally, in Section 2.6, we derive the probabilistic representation for the viscosity solution to the associated quadratic PDEs.

Chapter 3 addresses the second question by using a regularization procedure which is different from Morlais [26] and Mocha and Westray [25]. The first contribution is to obtain an existence and uniqueness result given a Lipschitz-continuous generator and a bounded integrand g . BSDEs of this type are called *Lipschitz-quadratic*, and serve as a basic ingredient for the study of quadratic BSDEs. In the second step, we prove a more general version of monotone stability result which allows one to construct solutions to quadratic BSDEs via Lipschitz-quadratic regularizations. Finally, we rely on a convexity assumption to obtain the uniqueness result via θ -technique.

Chapter 3 is organized as follows. Section 3.1 presents the basic notions of semimartingale BSDEs. Section 3.2 concerns the existence and uniqueness result for Lipschitz-quadratic BSDEs. In Section 3.3, we prove a general version of monotone stability. As an application, the existence of a bounded solution is immediate. In Section 3.4, existence and uniqueness of unbounded solution are proved. Finally, we show in Section that the martingale part of a solution defines an equivalent change of measure.

Chapter 4 is a survey of the stability result of quadratic semimartingales studied in Barrieu and El Karoui [4]. Section 4.2 introduces the notion of quadratic semimartingales and their characterizations. In Section 4.3, we use a forward point of view to address the

issue of convergence: the stability of quadratic semimartingales is proved in the first step; it is then used to deduce the convergence of the martingale parts. Finally, in Section 3.5, the solutions to quadratic BSDEs are characterized as quadratic semimartingales. As a counterpart, a corresponding monotone stability result for BSDEs are formulated. The prime advantage of this stability result, in contrast to others, is that the boundedness is no longer needed.

Chapter 2

$\mathbb{L}^p(p \geq 1)$ Solutions to Quadratic BSDEs

2.1 Preliminaries

In this chapter, we study a class of quadratic BSDEs driven by Brownian motion. We fix the time horizon $T > 0$ and a d -dimensional Brownian motion $(W_t)_{0 \leq t \leq T}$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. $(\mathcal{F}_t)_{0 \leq t \leq T}$ is the filtration generated by W and augmented by \mathbb{P} -null sets of \mathcal{F} . Any measurability will refer to this filtration. In particular, Prog denotes the progressive σ -algebra on $\Omega \times [0, T]$. Let us introduce the notion of BSDEs and their solutions in the following paragraph. As mentioned in the introduction, we exclusively study \mathbb{R} -valued BSDEs.

BSDEs: Definition and Solutions. Let ξ be an \mathbb{R} -valued \mathcal{F}_T -measurable random variable, $F : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ a $\text{Prog} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable random function. The BSDEs of our study can be written as

$$Y_t = \xi + \int_t^T F(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \quad (2.1)$$

where $\int_0^T Z_s dW_s$, sometimes denoted by $Z \cdot W$, refers to the vector stochastic integral; see, e.g., Shiryaev and Cherny [30]. We call a process (Y, Z) valued in $\mathbb{R} \times \mathbb{R}^d$ a *solution* to (2.1), if Y is a continuous adapted process and Z is a Prog -measurable process such that \mathbb{P} -a.s. $\int_0^T |Z_s|^2 ds < +\infty$ and $\int_0^T |F(s, Y_s, Z_s)| ds < +\infty$, and (2.1) holds \mathbb{P} -a.s. for any $t \in [0, T]$. The first inequality above ensures that Z is integrable with respect to W in the sense of vector stochastic integration. As a result, $Z \cdot W$ is a continuous local martingale. We call F the *generator*, ξ the *terminal value* and $(\xi, \int_0^T |F(s, 0, 0)| ds)$ the *data*. In our study, the integrability property of the data determines estimates for a solution. The conditions imposed on the generator are called the *structure conditions*. For notational convenience, we sometimes write (F, ξ) instead of (2.1) to denote the BSDE with generator F and terminal value ξ .

We are interested in BSDEs satisfying, \mathbb{P} -a.s. for any $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$,

$$\begin{aligned} \operatorname{sgn}(y)F(t, y, z) &\leq \alpha_t + \beta|y| + \gamma|z| + f(|y|)|z|^2, \\ |F(t, y, z)| &\leq \alpha_t + \varphi(|y|) + \gamma|z| + f(|y|)|z|^2, \end{aligned} \quad (2.2)$$

where \mathbb{P} -a.s. for any $t \in [0, T]$, $(y, z) \mapsto F(t, y, z)$ is continuous, α is an \mathbb{R}^+ -valued Prog-measurable process, $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a continuous nondecreasing function with $\varphi(0) = 0$, $f(|\cdot|) : \mathbb{R} \rightarrow \mathbb{R}$ is a measurable function and $\gamma \geq 0$. As will be seen later, the BSDEs satisfying (2.2) are solvable if $f(|\cdot|)$ belongs to \mathcal{I} , the set of integrable functions from \mathbb{R} to \mathbb{R} which are bounded on any compact subset of \mathbb{R} . Note that (2.2) has an even more general growth in y , compared to the assumption (1.2) which is studied by Bahlali et al [1].

Let us close this section by introducing all required notations for this chapter. For any random variable or process Y , we say Y has some property if this is true except on a \mathbb{P} -null subset of Ω . Hence we omit “ \mathbb{P} -a.s.” in situations without ambiguity. Define $\operatorname{sgn}(x) := \mathbb{I}_{\{x \neq 0\}} \frac{x}{|x|}$. For any càdlàg adapted process Y , set $Y_{s,t} := Y_t - Y_s$ and $Y^* := \sup_{t \in [0, T]} |Y_t|$. For any Prog-measurable process H , set $|H|_{s,t} := \int_s^t H_u du$ and $|H|_t := |H|_{0,t}$. \mathcal{T} stands for the set of stopping times valued in $[0, T]$ and \mathcal{S} denotes the space of continuous adapted processes. For any local martingale M , we call $\{\sigma_n\}_{n \in \mathbb{N}^+} \subset \mathcal{T}$ a *localizing sequence* if σ_n increases stationarily to T as n goes to $+\infty$ and $M_{\cdot \wedge \sigma_n}$ is a martingale for any $n \in \mathbb{N}^+$. For later use, we specify the following spaces under \mathbb{P} .

- \mathcal{S}^∞ : the set of bounded processes in \mathcal{S} ;
- $\mathcal{S}^p(p \geq 1)$: the set of $Y \in \mathcal{S}$ with $Y^* \in \mathbb{L}^p$;
- \mathcal{D} : the set of $Y \in \mathcal{S}$ such that $\{Y_\tau | \tau \in \mathcal{T}\}$ is uniformly integrable;
- \mathcal{M} : the space of \mathbb{R}^d -valued Prog-measurable processes Z such that \mathbb{P} -a.s. $\int_0^T |Z_s|^2 ds < +\infty$; for any $Z \in \mathcal{M}$, $Z \cdot W$ is a continuous local martingale;
- $\mathcal{M}^p(p > 0)$: the set of $Z \in \mathcal{M}$ with

$$\|Z\|_{\mathcal{M}^p} := \mathbb{E} \left[\left(\int_0^T |Z_s|^2 ds \right)^{\frac{p}{2}} \right]^{\frac{1}{p} \wedge 1} < +\infty;$$

in particular, \mathcal{M}^2 is a Hilbert space;

- $\mathcal{C}^p(\mathbb{R})$: the space of p times continuously differentiable functions from \mathbb{R} to \mathbb{R} ;
- $\mathcal{W}_{1,loc}^2(\mathbb{R})$: the Sobolev space of measurable maps $u : \mathbb{R} \rightarrow \mathbb{R}$ such that both u and its generalized derivatives u', u'' belong to $\mathbb{L}_{loc}^1(\mathbb{R})$.

The above spaces are Banach (respectively complete) under suitable norms (respectively metrics); we will not present these facts in more detail since they are not involved in our study. We call (Y, Z) a \mathbb{L}^p *solution* to (2.1) if $(Y, Z) \in \mathcal{S}^p \times \mathcal{M}^p$. This definition simply comes from the fact that its existence is ensured by data in \mathbb{L}^p . Analogously to most papers on \mathbb{R} -valued quadratic BSDEs, our existence result essentially relies on the monotone stability result of quadratic BSDEs; see, e.g., Kobylanski [22], Briand and Hu [9] or Section 3.3, Chapter 3.1.

2.2 Functions of Class \mathcal{I}

In this section, we introduce the basic ingredients used to treat the quadratic generator in (2.2). We recall that \mathcal{I} is the set of integrable functions from \mathbb{R} to \mathbb{R} which are bounded on any compact subset of \mathbb{R} .

u^f **Transform.** For any $f \in \mathcal{I}$, define $u^f : \mathbb{R} \rightarrow \mathbb{R}$ and M^f by

$$u^f(x) := \int_0^x \exp\left(2 \int_0^y f(u) du\right) dy,$$

$$M^f := \exp\left(2 \int_{-\infty}^{\infty} |f(u)| du\right).$$

Obviously, $1 \leq M^f < +\infty$. Moreover, the following properties hold by simple computations.

- (i) $u \in \mathcal{C}^1(\mathbb{R}) \cap \mathcal{W}_{1,loc}^2(\mathbb{R})$ and $u''(x) = 2f(x)u'(x)$ a.e.; if f is continuous, then $u \in \mathcal{C}^2(\mathbb{R})$;
- (ii) u is strictly increasing and bijective from \mathbb{R} to \mathbb{R} ;
- (iii) $u^{-1} \in \mathcal{C}^1(\mathbb{R}) \cap \mathcal{W}_{1,loc}^2(\mathbb{R})$; if f is continuous, then $u^{-1} \in \mathcal{C}^2(\mathbb{R})$;
- (iv) $\frac{|x|}{M} \leq |u(x)| \leq M|x|$ and $\frac{1}{M} \leq u'(x) \leq M$.

v^f **Transform.** For any $f \in \mathcal{I}$, define $v^f : \mathbb{R} \rightarrow \mathbb{R}^+$ by

$$v^f(x) := \int_0^{|x|} u^{(-f)}(y) \exp\left(2 \int_0^y f(u) du\right) dy.$$

Set $v := v^f$. Simple computations give

- (i) $v \in \mathcal{C}^1(\mathbb{R}) \cap \mathcal{W}_{1,loc}^2(\mathbb{R})$ and $v''(x) - 2f(|x|)|v'(x)| = 1$ a.e.; if f is continuous, then $v \in \mathcal{C}^2(\mathbb{R})$;
- (ii) $v(x) \geq 0$, $\text{sgn}(v'(x)) = \text{sgn}(x)$ and $v''(0) = 1$;

$$(iii) \quad \frac{x^2}{2M^2} \leq v(x) \leq \frac{M^2 x^2}{2} \text{ and } \frac{|x|}{M^2} \leq |v'(x)| \leq M^2 |x|.$$

In the sequel of our study, u^f and v^f exclusively stand for the above transforms associated with $f \in \mathcal{I}$. Hence in situations without ambiguity, we denote u^f, v^f, M^f by u, v, M , respectively.

2.3 Krylov Estimate and the Itô-Krylov Formula

The first auxiliary result is the Krylov estimate. Later, it is used to prove an Itô's-type formula for functions in $\mathcal{C}^1(\mathbb{R}) \cap \mathcal{W}_{1,loc}^2(\mathbb{R})$. This helps to deal with (possibly discontinuous) quadratic generators. As the second application, we derive a generalized Itô formula for $y \mapsto |y|^p$ ($p \geq 1$) which is not smooth enough for $1 \leq p < 2$. This is a basic tool to study \mathbb{L}^p ($p \geq 1$) solutions.

To allow the existence of a local time in particular situations, we study equations of type

$$Y_t = \xi + \int_t^T F(s, Y_s, Z_s) ds + \int_t^T dC_s - \int_t^T Z_s dW_s, \quad (2.3)$$

where C is a continuous adapted process of finite variation. We denote its total variation process by $V(C)$. Likewise, sometimes we denote (2.3) by (F, C, ξ) . The solution to (2.3) is defined analogously to that to (2.1).

Now we prove the Krylov estimate for (2.3). A more complicated version not needed for our study can be found in Bahlali et al [1].

Lemma 2.1 (Krylov Estimate) *For any measurable function $\psi : \mathbb{R} \rightarrow \mathbb{R}^+$,*

$$\mathbb{E} \left[\int_0^{\tau_m} \psi(Y_s) |Z_s|^2 ds \right] \leq 6m \|\psi\|_{\mathbb{L}^1([-m, m])}, \quad (2.4)$$

where τ_m is a stopping time defined by

$$\tau_m := \inf \left\{ t \geq 0 : |Y_t| + V_t(C) + \int_0^t |F(s, Y_s, Z_s)| ds \geq m \right\} \wedge T.$$

Proof. Without loss of generality we assume $\|\psi\|_{\mathbb{L}^1([-m, m])} < +\infty$. For each $n \in \mathbb{N}^+$, set

$$\tau_{m,n} := \tau_m \wedge \inf \left\{ t \geq 0 : \int_0^t |Z_s|^2 ds \geq n \right\}.$$

Let $a \in [-m, m]$. By Tanaka's formula,

$$\begin{aligned}
(Y_{t \wedge \tau_{m,n}} - a)^- &= (Y_0 - a)^- - \int_0^{t \wedge \tau_{m,n}} \mathbb{I}_{\{Y_s < a\}} dY_s + \frac{1}{2} L_{t \wedge \tau_{m,n}}^a(Y) \\
&= (Y_0 - a)^- + \int_0^{t \wedge \tau_{m,n}} \mathbb{I}_{\{Y_s < a\}} F(s, Y_s, Z_s) ds + \int_0^{t \wedge \tau_{m,n}} \mathbb{I}_{\{Y_s < a\}} dC_s \\
&\quad - \int_0^{t \wedge \tau_{m,n}} \mathbb{I}_{\{Y_s < a\}} Z_s dW_s + \frac{1}{2} L_{t \wedge \tau_{m,n}}^a(Y),
\end{aligned} \tag{2.5}$$

where $L^a(Y)$ is the local time of Y at a . To estimate the local time, we put it on the left-hand side and the rest terms on the right-hand side. Since $x \mapsto (x - a)^-$ is Lipschitz-continuous, we deduce from the definition of $\tau_{m,n}$ that

$$(Y_0 - a)^- - (Y_{t \wedge \tau_{m,n}} - a)^- \leq |Y_0 - Y_{t \wedge \tau_{m,n}}| \leq 2m.$$

Meanwhile, the definition of τ_m also implies that the sum of the ds -integral and dC -integral is bounded by m . Hence, we have

$$\mathbb{E}[L_{t \wedge \tau_{m,n}}^a(Y)] \leq 6m.$$

By Fatou's lemma applied to the sequence indexed by n ,

$$\sup_{a \in [-m, m]} \mathbb{E}[L_{t \wedge \tau_m}^a(Y)] \leq 6m.$$

We then use time occupation formula for continuous semimartingales (see Chapter VI., Revuz and Yor [29]) and the above inequality to obtain

$$\begin{aligned}
\mathbb{E}\left[\int_0^{T \wedge \tau_m} \psi(Y_s) |Z_s|^2 ds\right] &= \mathbb{E}\left[\int_{-m}^m \psi(x) L_{T \wedge \tau_m}^x(Y) dx\right] \\
&= \int_{-m}^m \psi(x) \mathbb{E}[L_{T \wedge \tau_m}^x(Y)] dx \\
&\leq 6m \|\psi\|_{\mathbb{L}^1([-m, m])}.
\end{aligned}$$

□

As an immediate consequence of Lemma 2.1, we have \mathbb{P} -a.s.

$$\int_0^T \mathbb{I}_{\{Y_s \in A\}} |Z_s|^2 ds = 0, \tag{2.6}$$

for any $A \subset \mathbb{R}$ with null Lebesgue measure. This will be used later several times.

Given Lemma 2.1, we turn to the main results of this section.

Theorem 2.2 (Itô-Krylov Formula) *If (Y, Z) is a solution to (F, C, ξ) , then for any $u \in \mathcal{C}^1(\mathbb{R}) \cap \mathcal{W}_{1,loc}^2(\mathbb{R})$, we have \mathbb{P} -a.s. for all $t \in [0, T]$,*

$$u(Y_t) = u(Y_0) + \int_0^t u'(Y_s) dY_s + \frac{1}{2} \int_0^t u''(Y_s) |Z_s|^2 ds. \quad (2.7)$$

Proof. We use τ_m defined in Lemma 2.1 (Krylov estimate). Note that τ_m increases stationarily to T as m goes to $+\infty$. It is therefore sufficient to prove the equality for $u(Y_{t \wedge \tau_m})$. To this end we use an approximation procedure. We consider m such that \mathbb{P} -a.s. $m \geq |Y_0|$. Let u_n be a sequence of functions in $\mathcal{C}^2(\mathbb{R})$ satisfying

- (i) u_n converges uniformly to u on $[-m, m]$;
- (ii) u'_n converges uniformly to u' on $[-m, m]$;
- (iii) u''_n converges in $\mathbb{L}^1([-m, m])$ to u'' .

By Itô's formula,

$$u_n(Y_{t \wedge \tau_m}) = u_n(Y_0) + \int_0^{t \wedge \tau_m} u'_n(Y_s) dY_s + \frac{1}{2} \int_0^{t \wedge \tau_m} u''_n(Y_s) |Z_s|^2 ds.$$

Due to (i) and $|Y_{t \wedge \tau_m}| \leq m$, $u_n(Y_{t \wedge \tau_m})$ converges to $u(Y_{t \wedge \tau_m})$ \mathbb{P} -a.s. uniformly on $[0, T]$ as n goes to $+\infty$; the second term converges in probability to

$$\int_0^{t \wedge \tau_m} u'(Y_s) dY_s$$

by (ii) and dominated convergence for stochastic integrals; the last term converges in probability to

$$\frac{1}{2} \int_0^{t \wedge \tau_m} u''(Y_s) |Z_s|^2 ds$$

due to (iii) and Lemma 2.1. Indeed, Lemma 2.1 implies

$$\mathbb{E} \left[\int_0^{\tau_m} |u''_n - u''|(Y_s) |Z_s|^2 ds \right] \leq 6m \|u''_n - u''\|_{\mathbb{L}^1([-m, m])}.$$

Hence collecting these convergence results gives (2.7). By the continuity of both sides of (2.7), the quality also holds \mathbb{P} -a.s. for all $t \in [0, T]$.

□

To study $\mathbb{L}^p(p \geq 1)$ solutions we now prove an Itô's-type formula for $y \mapsto |y|^p (p \geq 1)$ which is not smooth enough for $1 \leq p < 2$. The proof for multidimensional Itô processes can be found, e.g., in Briand et al [6]. In contrast to their approach, we give a novel and simpler proof for BSDE framework but point out that it can be also extended to Itô processes.

Lemma 2.3 *Let $p \geq 1$. If (Y, Z) is a solution to (F, C, ξ) , then*

$$\begin{aligned} |Y_t|^p + \frac{p(p-1)}{2} \int_t^T \mathbb{I}_{\{Y_s \neq 0\}} |Y_s|^{p-2} |Z_s|^2 ds \\ = |\xi|^p - p \int_t^T \operatorname{sgn}(Y_s) |Y_s|^{p-1} dY_s - \mathbb{I}_{\{p=1\}} \int_t^T dL_s^0(Y), \end{aligned} \quad (2.8)$$

where $L^0(Y)$ is the local time of Y at 0.

Proof. (i). $p = 1$. This is immediate from Tanaka's formula.

(ii). $p > 2$. $y \mapsto |y|^p \in \mathcal{C}^2(\mathbb{R})$. Hence this is immediate from Itô's formula.

(iii). $p = 2$. $y \mapsto |y|^p \in \mathcal{C}^2(\mathbb{R})$. Due to (2.6), $\int_0^\cdot |Y_s|^{p-2} |Z_s|^2 ds$ is indistinguishable from $\int_0^\cdot \mathbb{I}_{\{Y_s \neq 0\}} |Y_s|^{p-2} |Z_s|^2 ds$. Then the inequality is immediate from Itô's formula.

(iv). $1 < p < 2$. We use an approximation argument. Define

$$u_\epsilon(y) := (y^2 + \epsilon^2)^{\frac{1}{2}}.$$

Then for any $\epsilon > 0$, we have $u_\epsilon^p \in \mathcal{C}^2(\mathbb{R})$. By Itô's formula,

$$u_\epsilon^p(Y_t) = u_\epsilon^p(\xi) - p \int_t^T Y_s u_\epsilon^{p-2}(Y_s) dY_s - \frac{1}{2} \int_t^T (p u_\epsilon^{p-2}(Y_s) + p(p-2) |Y_s|^2 u_\epsilon^{p-4}(Y_s)) |Z_s|^2 ds. \quad (2.9)$$

Now we send ϵ to 0. $u_\epsilon(y) \rightarrow |y|$ pointwise implies $u_\epsilon(Y_t)^p \rightarrow |Y_t|^p$ and $u_\epsilon(\xi)^p \rightarrow |\xi|^p$ pointwise on Ω . Secondly, $y u_\epsilon^{p-2}(y) \rightarrow \operatorname{sgn}(y) |y|^{p-1}$ pointwise implies by dominated convergence for stochastic integrals that

$$\int_t^T Y_s \operatorname{sgn}(Y_s) u_\epsilon^{p-2}(Y_s) dY_s \rightarrow \int_t^T |Y_s|^{p-1} dY_s \text{ in probability.}$$

To prove that the ds -integral in (2.9) also converges, we split it into two parts and argue their convergence respectively. Note that

$$p u_\epsilon^{p-2}(Y_s) + p(p-2) |Y_s|^2 u_\epsilon^{p-4}(Y_s) = p \epsilon^2 u_\epsilon^{p-4}(Y_s) + p(p-1) |Y_s|^2 u_\epsilon^{p-4}(Y_s). \quad (2.10)$$

For the second term on the right-hand side of (2.10), we have

$$|Y_s|^2 u_\epsilon^{p-4}(Y_s) = \mathbb{I}_{\{Y_s \neq 0\}} |Y_s|^{p-2} \left| \frac{|Y_s|}{u_\epsilon(Y_s)} \right|^{4-p}.$$

Since $\frac{|y|}{u_\epsilon(y)} \nearrow \mathbb{I}_{\{y \neq 0\}}$, monotone convergence gives

$$\int_t^T |Y_s|^2 u_\epsilon^{p-4}(Y_s) |Z_s|^2 ds \rightarrow \int_t^T \mathbb{I}_{\{Y_s \neq 0\}} |Y_s|^{p-2} |Z_s|^2 ds \text{ pointwise in } \Omega.$$

It thus remains to prove the ds -integral concerning the first term on the right-hand side of (2.10) converges to 0. To this end, we use Lemma 2.1 (Krylov estimate) and the same localization procedure. This gives

$$\begin{aligned}
\mathbb{E} \left[\int_0^{\tau_m} \epsilon^2 u_\epsilon^{p-4}(Y_s) |Z_s|^2 ds \right] &\leq 6m\epsilon^2 \int_{-m}^m (x^2 + \epsilon^2)^{\frac{p-4}{2}} dx \\
&\leq 12m\epsilon^2 \int_0^m (x^2 + \epsilon^2)^{\frac{p-4}{2}} dx \\
&\leq 12 \cdot 2^{\frac{4-p}{2}} m\epsilon^2 \int_0^m (x + \epsilon)^{p-4} dx \\
&\leq 12 \cdot 2^{\frac{4-p}{2}} m\epsilon^2 \int_\epsilon^{m+\epsilon} x^{p-4} dx \\
&= \frac{12 \cdot 2^{\frac{4-p}{2}} m}{p-3} (\epsilon^2(m+\epsilon)^{p-3} - \epsilon^{p-1}),
\end{aligned}$$

which, due to $1 < p$, converges to 0 as ϵ goes to 0. Hence $\int_0^\cdot \epsilon^2 u_\epsilon^{p-4}(Y_s) |Z_s|^2 ds$ converges *u.c.p* to 0. Collecting all convergence results above gives (2.8). By the continuity of each term in (2.8), the equality also holds \mathbb{P} -a.s. for all $t \in [0, T]$.

□

2.4 $\mathbb{L}^p(p \geq 1)$ Solutions to Purely Quadratic BSDEs

Before turning to the main results, we partially extend the existence and uniqueness result for purely quadratic BSDEs studied by Bahlali et al [1]. Later, we present their natural extensions and the motivations of our work. These BSDEs are called purely quadratic, since the generator takes the form $F(t, y, z) = f(y)|z|^2$. The solvability simply comes from the function u^f defined in Section 2.2 which transforms better known BSDEs to $(f(y)|z|^2, \xi)$ by Itô-Krylov formula.

Theorem 2.4 *Let $f \in \mathcal{I}$ and $\xi \in \mathbb{L}^p(p \geq 1)$. Then there exists a unique solution to*

$$Y_t = \xi + \int_t^T f(Y_s) |Z_s|^2 ds - \int_t^T Z_s dW_s. \quad (2.11)$$

Moreover, if $p > 1$, the solution belongs to $\mathcal{S}^p \times \mathcal{M}^p$; if $p = 1$, the solution belongs to $\mathcal{D} \times \mathcal{M}^q$ for any $q \in (0, 1)$.

Proof. Let $u := u^f$. Then $u, u^{-1} \in \mathcal{C}^1(\mathbb{R}) \cap \mathcal{W}_{1,loc}^2(\mathbb{R})$. The existence and uniqueness result can be seen as a one-on-one correspondence between solutions to BSDEs.

(i). Existence. $|u(x)| \leq M|x|$ implies $u(\xi) \in \mathbb{L}^p$. By Itô representation theorem, there exists a unique pair (\tilde{Y}, \tilde{Z}) which solves $(0, u(\xi))$, i.e.,

$$d\tilde{Y}_t = \tilde{Z}_t dW_t, \quad \tilde{Y}_T = u(\xi). \quad (2.12)$$

We aim at proving

$$(Y, Z) := (u^{-1}(\tilde{Y}), \frac{\tilde{Z}}{u'(u^{-1}(\tilde{Y}))}) \quad (2.13)$$

solves (2.11). Itô-Krylov formula applied to $Y_t = u^{-1}(\tilde{Y}_t)$ yields

$$dY_t = \frac{1}{u'(u^{-1}(\tilde{Y}_t))} d\tilde{Y}_t - \frac{1}{2} \left(\frac{1}{u'(u^{-1}(\tilde{Y}_t))} \right)^2 \frac{u''(u^{-1}(\tilde{Y}_t))}{u'(u^{-1}(\tilde{Y}_t))} |\tilde{Z}_s|^2 ds. \quad (2.14)$$

To simplify (2.14) let us recall that $u''(x) = 2f(x)u'(x)$ a.e. Hence (2.13), (2.14) and (2.6) give

$$dY_t = -f(Y_t)|Z_t|^2 dt + Z_t dW_t, \quad Y_T = \xi,$$

i.e., (Y, Z) solves (2.11).

(ii). Uniqueness. Suppose (Y, Z) and (Y', Z') are solutions to (2.11). By Itô-Krylov formula applied to $u(Y)$ and $u(Y')$, we deduce that $(u(Y), u'(Y)Z)$ and $(u(Y'), u'(Y')Z')$ solve $(0, u(\xi))$. But from (i) it is known that they coincide. Transforming $u(Y)$ and $u(Y')$ via the bijective function u^{-1} yields the uniqueness result.

(iii). We prove the estimate for the unique solution (Y, Z) . For $p > 1$, Doob's $\mathbb{L}^p(p > 1)$ maximal inequality used to (2.12) implies $(\tilde{Y}, \tilde{Z}) \in \mathcal{S}^p \times \mathcal{M}^p$. Hence $(Y, Z) \in \mathcal{S}^p \times \mathcal{M}^p$, due to $|u'(x)| \geq \frac{1}{M}$ and $|u^{-1}(x)| \leq M|x|$. For $p = 1$, $\tilde{Y} \in \mathcal{D}$ since it is a martingale on $[0, T]$. In view of the above properties of u we have $Y \in \mathcal{D}$. The estimate for Z is immediate from Lemma 6.1, Briand et al [6] which is a version of $\mathbb{L}^p(0 < p < 1)$ maximal inequality for martingales.

□

Remark. If ξ is a general \mathcal{F}_T -measurable random variable, Dudley representation theorem (see Dudley [14]) implies that there still exists a solution to (2.12) and hence a solution to (2.11). However, the solution in general is not unique.

The proof of Theorem 2.4 indicates that f being bounded on compact subsets of \mathbb{R} is not needed for the existence and uniqueness result of purely quadratic BSDEs.

Proposition 2.5 (Comparison) *Let $f, g \in \mathcal{I}$, $\xi, \xi' \in \mathbb{L}^p(p \geq 1)$ and $(Y, Z), (Y', Z')$ be the unique solutions to $(f(y)|z|^2, \xi), (g(y)|z|^2, \xi')$, respectively. If $f \leq g$ a.e. and \mathbb{P} -a.s. $\xi \leq \xi'$, then \mathbb{P} -a.s. $Y \leq Y'$.*

Proof. Again we transform so as to compare better known BSDEs. Let us fix $t \in [0, T]$ and set $u := u^f$. For any stopping time valued in $[t, T]$, Itô-Krylov formula yields

$$\begin{aligned} u(Y'_t) &= u(Y'_\tau) + \int_t^\tau \left(u'(Y'_s)g(Y'_s)|Z'_s|^2 - \frac{1}{2}u''(Y'_s)|Z'_s|^2 \right) ds - \int_t^\tau u'(Y_s)Z'_s dW_s. \\ &= u(Y'_\tau) + \int_t^\tau u'(Y'_s)(g(Y'_s) - f(Y'_s))|Z'_s|^2 ds - \int_t^\tau u'(Y_s)Z'_s dW_s \\ &\geq u(Y'_\tau) - \int_t^\tau u'(Y_s)Z'_s dW_s, \end{aligned}$$

where the last two lines are due to $u''(x) = 2f(x)u'(x)$ a.e., $g \geq f$ a.e. and (2.6). In the next step, we want to eliminate the local martingale part by a localization procedure. Note that $\int_t^\tau u'(Y_s)Z'_s dW_s$ is a local martingale on $[t, T]$. Set $\{\tau_n\}_{n \in \mathbb{N}^+}$ to be its localizing sequence on $[t, T]$. Replacing τ by τ_n in the above inequality thus gives \mathbb{P} -a.s.

$$u(Y'_t) \geq \mathbb{E}[u(Y'_{t \wedge \tau_n}) | \mathcal{F}_t].$$

This implies that, for any $A \in \mathcal{F}_t$, we have

$$\mathbb{E}[u(Y'_t)\mathbb{I}_A] \geq \mathbb{E}[u(Y'_{t \wedge \tau_n})\mathbb{I}_A].$$

Since $u(Y') \in \mathcal{D}$, we can use Vitali convergence theorem to obtain

$$\mathbb{E}[u(Y'_t)\mathbb{I}_A] \geq \mathbb{E}[u(\xi')\mathbb{I}_A] = \mathbb{E}[\mathbb{E}[u(\xi') | \mathcal{F}_t]\mathbb{I}_A].$$

Note that this inequality holds for any $A \in \mathcal{F}_t$. Hence, by choosing $A = \{u(Y'_t) < \mathbb{E}[u(\xi') | \mathcal{F}_t]\}$, we obtain $u(Y'_t) \geq \mathbb{E}[u(\xi') | \mathcal{F}_t]$. Since $\xi' \geq \xi$ and u is increasing, we further have $u(Y'_t) \geq \mathbb{E}[u(\xi) | \mathcal{F}_t]$. Let us recall that, by Theorem 2.4, $(u(Y), u'(Y)Z)$ is the unique solution to $(0, u(\xi))$. Hence, $u(Y'_t) \geq u(Y_t)$. Transforming both sides via the bijective increasing function u^{-1} yields \mathbb{P} -a.s. $Y_t \leq Y'_t$. By the continuity of Y and Y' we have \mathbb{P} -a.s. $Y \leq Y'$.

□

Remark. In Proposition 2.5, we rely on the fact that \mathbb{P} -a.s.

$$\int_0^\cdot \left(\frac{1}{2}u''(Y'_s) - f(Y'_s)u'(Y'_s) \right) |Z'_s|^2 ds = 0, \quad (2.15)$$

even though $u''(x) = 2f(x)u'(x)$ only holds almost everywhere on \mathbb{R} . Here we prove it. Let A be the subset of \mathbb{R} on which $u''(x) = 2f(x)u'(x)$ fails. Hence,

$$\int_0^\cdot \mathbb{I}_{\{Y'_s \in \mathbb{R} \setminus A\}} \left| \frac{1}{2}u''(Y'_s) - f(Y'_s)u'(Y'_s) \right| |Z'_s|^2 ds = 0.$$

Meanwhile, by (2.6), we have \mathbb{P} -a.s.

$$\int_0^\cdot \mathbb{I}_{\{Y'_s \in A\}} \left| \frac{1}{2} u''(Y'_s) - f(Y'_s) u'(Y'_s) \right| |Z'_s|^2 ds = 0.$$

Hence, (2.15) holds \mathbb{P} -a.s. This fact also applies to Theorem 2.4 and all results in the sequel of our study.

To end our discussions on purely quadratic BSDEs we give some examples.

Example 2.6 Let $\xi \in \mathbb{L}^p(p \geq 1)$. Then Theorem 2.4 holds for (F, ξ) , where F verifies any one of the following

- $F(y, z) = \sin(y) \mathbb{I}_{[-\pi, \frac{\pi}{2}]}(y) |z|^2$;
- $F(y, z) = (\mathbb{I}_{[a, b]} - \mathbb{I}_{[c, d]})(y) |z|^2$ for some $a < b$ and $c < d$;
- $F(y, z) = \mathbb{I}_{\{y \neq 0\}} \frac{1}{(1+y^2)\sqrt{|y|}} |z|^2 + \mathbb{I}_{\{y=0\}} |z|^2$.

Theorem 2.4 and Proposition 2.5 are based on a one-on-one correspondence between solutions (respectively the unique solution) to BSDEs. Hence it is natural to generalize as follows. Set $f \in \mathcal{I}$, $u := u^f$, $F(t, y, z) := G(t, y, z) + f(y) |z|^2$ and

$$\tilde{F}(t, y, z) := u'(u^{-1}(y)) G(t, u^{-1}(y), \frac{z}{u'(u^{-1}(y))}). \quad (2.16)$$

If G ensures the existence of a solution to $(\tilde{F}, u(\xi))$, we can transform it via u^{-1} to a solution to (F, ξ) . An example is that G is of continuous linear growth in (y, z) where the existence of a maximal (respectively minimal) solution to $(\tilde{F}, u(\xi))$ can be proved in the spirit of Lepeltier and San Martin [23].

When the generator is continuous in (y, z) , a more general situation is linear-quadratic growth, i.e.,

$$|H(t, y, z)| \leq \alpha_t + \beta |y| + \gamma |z| + f(|y|) |z|^2 := F(t, y, z). \quad (2.17)$$

The existence result then consists of viewing the maximal (respectively minimal) solution to (F, ξ^+) (respectively $(-F, -\xi^-)$) as a priori bounds for solutions to (H, ξ) , and using a combination of a localization procedure and the monotone stability result developed by Briand and Hu [8], [9]. For details the reader shall refer to Bahlali et al [1].

However, either an additive structure in (2.16) or a linear-quadratic growth (2.17) is too restrictive and uniqueness is not available in general. Considering this limitation, we devote Section 2.5 to the solvability under milder structure conditions.

2.5 $\mathbb{L}^p(p > 1)$ Solutions to Quadratic BSDEs

With the preparatory work in Section 2.1, 2.2, 2.3, 2.4, we study $\mathbb{L}^p(p > 1)$ solutions to quadratic BSDEs under general assumptions. We deal with the quadratic generators in the spirit of Bahlali et al [1], derive the estimates in the spirit of Briand et al [6] and prove the existence and uniqueness result in the spirit of Briand et al [8], [9], [10]. This section can also be seen as a generalization of these works. The following assumptions on (F, ξ) ensure the a priori estimates and an existence result.

Assumption (A.1) Let $p \geq 1$. There exist $\beta \in \mathbb{R}, \gamma \geq 0$, an \mathbb{R}^+ -valued Prog-measurable process $\alpha, f(|\cdot|) \in \mathcal{I}$ and a continuous nondecreasing function $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $\varphi(0) = 0$ such that $|\xi| + |\alpha|_T \in \mathbb{L}^p$ and \mathbb{P} -a.s.

(i) for any $t \in [0, T]$, $(y, z) \mapsto F(t, y, z)$ is continuous;

(ii) F is “monotonic” at $y = 0$, i.e., for any $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$,

$$\operatorname{sgn}(y)F(t, y, z) \leq \alpha_t + \beta|y| + \gamma|z| + f(|y|)|z|^2;$$

(iii) for any $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$,

$$|F(t, y, z)| \leq \alpha_t + \varphi(|y|) + \gamma|z| + f(|y|)|z|^2.$$

It is worth noticing that given (A.1)(iii) and $f = 0$, (A.1)(ii) is a consequence of F being monotonic at $y = 0$. Indeed,

$$\operatorname{sgn}(y - 0)(F(t, y, z) - F(t, 0, z)) \leq \beta|y|$$

implies

$$\begin{aligned} \operatorname{sgn}(y)F(t, y, z) &\leq F(t, 0, z) + \beta|y| \\ &\leq \alpha_t + \beta|y| + \gamma|z|. \end{aligned}$$

This explains why we keep saying that F is monotonic at $y = 0$, even though y also appears in the quadratic term. Secondly, our results don’t rely on the specific choice of φ . Hence the growth condition in y can be arbitrary as long as (A.1)(i)(ii) hold. Assumptions of this type for different settings can also be found in, e.g., [27], [7], [6], [10], [9]. Finally, f can be discontinuous; $f(|\cdot|)$ being \mathbb{R}^+ -valued appears more naturally in the growth condition.

Lemma 2.7 (A Priori Estimate (i)) *Let $p \geq 1$ and (A.1) hold for (F, ξ) . If $(Y, Z) \in \mathcal{S}^p \times \mathcal{M}$ is a solution to (F, ξ) , then*

$$\mathbb{E}\left[\left(\int_0^T |Z_s|^2 ds\right)^{\frac{p}{2}}\right] + \mathbb{E}\left[\left(\int_0^T f(|Y_s|)|Z_s|^2 ds\right)^p\right] \leq c\left(\mathbb{E}[(Y^*)^p + |\alpha|_T^p]\right),$$

where c is a constant only depending on $T, M^{f(|\cdot|)}, \beta, \gamma, p$.

Proof. Set $v := v^{f(\cdot|\cdot)}$ and $M := M^{f(\cdot|\cdot)}$. For any $\tau \in \mathcal{T}$, Itô-Krylov formula yields

$$\begin{aligned} v(Y_0) &= v(Y_\tau) + \int_0^\tau v'(Y_s)F(s, Y_s, Z_s)ds \\ &\quad - \frac{1}{2} \int_0^\tau v''(Y_s)|Z_s|^2ds - \int_0^\tau v'(Y_s)Z_s dW_s. \end{aligned} \quad (2.18)$$

Due to $\text{sgn}(v'(x)) = \text{sgn}(x)$ and (A.1)(ii), we have

$$v'(Y_s)F(s, Y_s, Z_s) \leq |v'(Y_s)|(\alpha_s + \beta|Y_s| + \gamma|Z_s| + f(|Y_s|)|Z_s|^2). \quad (2.19)$$

Recall that $v''(x) - 2f(|x|)|v'(x)| = 1$ a.e. Hence (2.18), (2.19) and (2.6) give

$$\frac{1}{2} \int_0^\tau |Z_s|^2ds \leq v(Y_\tau) + \int_0^\tau |v'(Y_s)|(\alpha_s + \beta|Y_s| + \gamma|Z_s|)ds - \int_0^\tau v'(Y_s)Z_s dW_s.$$

Moreover, since $v(x) \leq \frac{M^2x^2}{2}$ and $|v'(x)| \leq M^2|x|$, this inequality gives

$$\int_0^\tau |Z_s|^2ds \leq c_1(Y^*)^2 + c_1 \int_0^\tau |Y_s|(\alpha_s + |Y_s| + |Z_s|)ds - 2 \int_0^\tau v'(Y_s)Z_s dW_s, \quad (2.20)$$

where $c_1 := 2M^2(1 \vee \beta \vee \gamma)$. Note that in (2.20),

$$\begin{aligned} \int_0^\tau |Y_s|\alpha_s ds &\leq \frac{1}{2}(Y^*)^2 + \frac{1}{2}|\alpha|_T^2, \\ c_1 \int_0^\tau |Y_s||Z_s|ds &\leq \frac{1}{2}c_1^2T \cdot (Y^*)^2 + \frac{1}{2} \int_0^\tau |Z_s|^2ds. \end{aligned}$$

Hence (2.20) yields

$$\int_0^\tau |Z_s|^2ds \leq (3c_1 + c_1^2T)(Y^*)^2 + c_1|\alpha|_T^2 - 4 \int_0^\tau v'(Y_s)Z_s dW_s.$$

This estimate implies that for any $p \geq 1$,

$$\mathbb{E}\left[\left(\int_0^\tau |Z_s|^2ds\right)^{\frac{p}{2}}\right] \leq c_2\mathbb{E}\left[(Y^*)^p + |\alpha|_T^p + \left|\int_0^\tau v'(Y_s)Z_s dW_s\right|^{\frac{p}{2}}\right], \quad (2.21)$$

where $c_2 := 3^{\frac{p}{2}}((3c_1 + c_1^2T) \vee 4)^{\frac{p}{2}}$. Define for each $n \in \mathbb{N}^+$, $\tau_n := \inf\{t \geq 0 : \int_0^t |Z_s|^2ds \geq n\} \wedge T$. We then replace τ by τ_n and use Davis-Burkholder-Gundy inequality to obtain

$$\begin{aligned} c_2\mathbb{E}\left[\left(\int_0^{\tau_n} v'(Y_s)Z_s dW_s\right)^{\frac{p}{2}}\right] &\leq c_2c(p)M^p\mathbb{E}\left[\left(\int_0^{\tau_n} |Y_s|^2|Z_s|^2ds\right)^{\frac{p}{4}}\right] \\ &\leq \frac{1}{2}c_2^2c(p)^2M^{2p} \cdot \mathbb{E}[(Y^*)^p] + \frac{1}{2}\mathbb{E}\left[\left(\int_0^{\tau_n} |Z_s|^2ds\right)^{\frac{p}{2}}\right] \\ &< +\infty. \end{aligned}$$

We explain that in this inequality, $c(p)$ denotes the constant in Davis-Burkholder-Gundy inequality which only depends on p . With this estimate, we come back to (2.21). Transferring the quadratic term to the left-hand side of (2.21) and using Fatou's lemma, we obtain

$$\mathbb{E}\left[\left(\int_0^T |Z_s|^2 ds\right)^{\frac{p}{2}}\right] \leq c\left(\mathbb{E}[(Y^*)^p + |\alpha|_T^p]\right),$$

where $c := c_2^2 c(p)^2 M^{2p} + 2c_2$.

To estimate $\int_0^T f(|Y_s|)|Z_s|^2 ds$ we use $u := u^{2f(\cdot)}$. This helps to transfer $\int_0^T f(|Y_s|)|Z_s|^2 ds$ to the left-hand side so that standard estimates can be used. The proof is omitted since it is not relevant to our study.

□

We continue our study by sharpening Lemma 2.7 for $p > 1$. We follow Proposition 3.2, Briand et al [6] and extend it to quadratic BSDEs. As an important byproduct, we obtain the a priori bound for solutions which is crucial to the construction of a solution.

Lemma 2.8 (A Priori Estimate (ii)) *Let $p > 1$ and (A.1) hold for (F, ξ) . If $(Y, Z) \in \mathcal{S}^p \times \mathcal{M}$ is a solution to (F, ξ) , then*

$$\mathbb{E}[(Y^*)^p] + \mathbb{E}\left[\left(\int_0^T |Z_s|^2 ds\right)^{\frac{p}{2}}\right] + \mathbb{E}\left[\left(\int_0^T f(|Y_s|)|Z_s|^2 ds\right)^p\right] \leq c\left(\mathbb{E}[|\xi|^p + |\alpha|_T^p]\right).$$

In particular,

$$\mathbb{E}\left[\sup_{s \in [t, T]} |Y_s|^p \middle| \mathcal{F}_t\right] \leq c\mathbb{E}[|\xi|^p + |\alpha|_{t, T}^p | \mathcal{F}_t].$$

In both cases, c is a constant only depending on $T, M^{f(\cdot)}, \beta, \gamma, p$.

Proof. Let $u := u^{f(\cdot)}$ and $M := M^{f(\cdot)}$, and denote $u(|Y_t|), u'(|Y_t|), u''(|Y_t|)$ by u_t, u'_t, u''_t , respectively. By Tanaka's formula applied to $|Y_t|$ and Itô-Krylov formula applied to u_t ,

$$\begin{aligned} u_t &= u_T + \int_t^T \operatorname{sgn}(Y_s) u'_s F(s, Y_s, Z_s) ds - \frac{1}{2} \int_t^T \mathbb{I}_{\{Y_s \neq 0\}} u''_s |Z_s|^2 ds \\ &\quad - \int_t^T \operatorname{sgn}(Y_s) u'_s Z_s dW_s - \int_t^T u'_s dL_s^0(Y), \end{aligned}$$

where $L^0(Y)$ is the local time of Y at 0. Lemma 2.3 applied to u_t then gives

$$\begin{aligned} |u_t|^p &+ \frac{p(p-1)}{2} \int_t^T \mathbb{I}_{\{u_s \neq 0\}} \mathbb{I}_{\{Y_s \neq 0\}} |u_s|^{p-2} |u'_s|^2 |Z_s|^2 ds \\ &= |u_T|^p + p \int_t^T \operatorname{sgn}(u_s) |u_s|^{p-1} \left(\operatorname{sgn}(Y_s) u'_s F(s, Y_s, Z_s) - \frac{1}{2} \mathbb{I}_{\{Y_s \neq 0\}} u''_s |Z_s|^2 \right) ds \\ &\quad - p \int_t^T \operatorname{sgn}(u_s) |u_s|^{p-1} u'_s dL_s^0(Y) - p \int_t^T \operatorname{sgn}(u_s) \operatorname{sgn}(Y_s) |u_s|^{p-1} u'_s Z_s dW_s. \end{aligned}$$

To simplify this equality we recall that $\text{sgn}(u_s) = \mathbb{I}_{\{u_s \neq 0\}} = \mathbb{I}_{\{Y_s \neq 0\}}$ and $u''(x) = 2f(x)u'(x)$ a.e. Hence

$$\begin{aligned} & |u_t|^p + \frac{p(p-1)}{2} \int_t^T \mathbb{I}_{\{Y_s \neq 0\}} |u_s|^{p-2} |u'_s|^2 |Z_s|^2 ds \\ & \leq |u_T|^p + p \int_t^T \mathbb{I}_{\{Y_s \neq 0\}} |u_s|^{p-1} u'_s (\alpha_s + \beta |Y_s| + \gamma |Z_s|) ds \\ & \quad - p \int_t^T \text{sgn}(Y_s) |u_s|^{p-1} u'_s Z_s dW_s. \end{aligned}$$

Let $\{c_n\}_{n \in \mathbb{N}^+}$ be constants to be determined. Since $\frac{|x|}{M} \leq u(|x|) \leq M|x|$ and $\frac{1}{M} \leq u'(|x|) \leq M$, this inequality yields

$$\begin{aligned} & |Y_t|^p + c_1 \int_t^T \mathbb{I}_{\{Y_s \neq 0\}} |Y_s|^{p-2} |Z_s|^2 ds \\ & \leq M^p |\xi|^p + M^p \int_t^T \mathbb{I}_{\{Y_s \neq 0\}} |Y_s|^{p-1} (\alpha_s + |\beta| |Y_s| + \gamma |Z_s|) ds \\ & \quad - p \int_t^T \text{sgn}(Y_s) |u_s|^{p-1} u'_s Z_s dW_s, \end{aligned} \tag{2.22}$$

where $c_1 := \frac{p(p-1)}{2M^p} > 0$. Observe that in (2.22),

$$M^p \gamma \mathbb{I}_{\{Y_s \neq 0\}} |Y_s|^{p-1} |Z_s| \leq \frac{M^{2p} \gamma^2}{2c_1} |Y_s|^p + \frac{c_1}{2} \mathbb{I}_{\{Y_s \neq 0\}} |Y_s|^{p-2} |Z_s|^2.$$

We then use this inequality to (2.22). Set $c_2 := M^p \vee (M^p \beta + \frac{M^{2p} \gamma^2}{2c_1})$,

$$X := c_2 \left(|\xi|^p + \int_0^T |Y_s|^{p-1} (\alpha_s + |Y_s|) ds \right),$$

and N to be the local martingale part of (2.22). Hence (2.22) gives

$$|Y_t|^p + \frac{c_1}{2} \int_t^T \mathbb{I}_{\{Y_s \neq 0\}} |Y_s|^{p-2} |Z_s|^2 ds \leq X - N_T + N_t. \tag{2.23}$$

We claim that N is a martingale. Let $c(1)$ be the constant in Davis-Burkholder-Gundy inequality for $p = 1$. We have

$$\begin{aligned} \mathbb{E}[N^*] & \leq c(1) \mathbb{E}[\langle N \rangle_T^{\frac{1}{2}}] \\ & \leq c(1) M^p \mathbb{E} \left[\left(\int_0^T |Y_s|^{2p-2} |Z_s|^2 ds \right)^{\frac{1}{2}} \right] \\ & \leq \frac{c(1) M^p}{p} \left((p-1) \mathbb{E}[(Y^*)^p] + \mathbb{E} \left[\left(\int_0^T |Z_s|^2 ds \right)^{\frac{p}{2}} \right] \right) \\ & < +\infty, \end{aligned}$$

where the last two lines come from Young's inequality and Lemma 2.7 (a priori estimate (i)). Hence N is a martingale. Coming back to (2.23), we deduce that

$$\mathbb{E}\left[\int_0^T \mathbb{I}_{\{Y_s \neq 0\}} |Y_s|^{p-2} |Z_s|^2 ds\right] \leq \frac{2}{c_1} \mathbb{E}[X]. \quad (2.24)$$

Now we estimate Y via X . To this end, taking supremum over $t \in [0, T]$ and using Davis-Burkholder-Gundy inequality to (2.23) give

$$\mathbb{E}[(Y^*)^p] \leq \mathbb{E}[X] + c(1)\mathbb{E}[\langle N \rangle_T^{\frac{1}{2}}]. \quad (2.25)$$

Here $c(1)$ denotes the constant in Davis-Burkholder-Gundy inequality for $p = 1$. The second term in (2.25) yields by Cauchy-Schwartz inequality that

$$\begin{aligned} c(1)\mathbb{E}[\langle N \rangle_T^{\frac{1}{2}}] &\leq c(1)M^p \mathbb{E}\left[(Y^*)^{\frac{p}{2}} \left(\int_0^T \mathbb{I}_{\{Y_s \neq 0\}} |Y_s|^{p-2} |Z_s|^2 ds\right)^{\frac{1}{2}}\right] \\ &\leq \frac{1}{2}\mathbb{E}[(Y^*)^p] + \frac{c(1)^2 M^{2p}}{2} \mathbb{E}\left[\int_0^T \mathbb{I}_{\{Y_s \neq 0\}} |Y_s|^{p-2} |Z_s|^2 ds\right]. \end{aligned}$$

Using (2.24) to this inequality gives the estimate of $\langle N \rangle_T^{\frac{1}{2}}$ via Y and X . With this estimate we come back to (2.25) and obtain

$$\mathbb{E}[(Y^*)^p] \leq 2\left(1 + \frac{2c(1)^2 M^{2p}}{c_1}\right) \mathbb{E}[X].$$

Set $c_3 := 2c_2\left(1 + \frac{c(1)^2 M^{2p}}{2}\right)$. This inequality yields

$$\mathbb{E}[(Y^*)^p] \leq c_3 \left(\mathbb{E}[|\xi|^p] + \mathbb{E}\left[\int_0^T |Y_s|^{p-1} \alpha_s ds\right] + \mathbb{E}\left[\int_0^T |Y_s|^p ds\right] \right). \quad (2.26)$$

Young's inequality used to the second term on the right-hand side of this inequality gives

$$c_3 \int_0^T |Y_s|^{p-1} \alpha_s ds \leq \frac{1}{2} (Y^*)^p + \frac{c_3}{p} \left(\frac{2}{c_3 q}\right)^{\frac{p}{q}} |\alpha|_T^p,$$

where q is the conjugate index of p . Set $c_4 := 2\left(c_3 \vee \frac{c_3}{p} \left(\frac{2}{c_3 q}\right)^{\frac{p}{q}}\right)$. (2.26) and this inequality then yield

$$\mathbb{E}[(Y^*)^p] \leq c_4 \left(\mathbb{E}[|\xi|^p + |\alpha|_T^p] + \mathbb{E}\left[\int_0^T \sup_{u \in [0, s]} |Y_u|^p ds\right] \right),$$

By Gronwall's lemma,

$$\mathbb{E}[(Y^*)^p] \leq c_4 \exp(c_4 T) \mathbb{E}[|\xi|^p + |\alpha|_T^p].$$

Finally by Lemma 2.7 we conclude that there exists a constant c only depending on T, M, β, γ, p such that

$$\mathbb{E}[(Y^*)^p] + \mathbb{E}\left[\left(\int_0^T |Z_s|^2 ds\right)^{\frac{p}{2}}\right] + \mathbb{E}\left[\left(\int_0^T f(|Y_s|)|Z_s|^2 ds\right)^p\right] \leq c\mathbb{E}[|\xi|^p + |\alpha|_T^p].$$

To prove the remaining statement, we view any fixed $t \in [0, T]$ as the initial time, reset

$$X := c_2\left(|\xi|^p + \int_t^T |Y_s|^{p-1}(\alpha_s + |Y_s|)ds\right)$$

and replace all estimates by conditional estimates.

□

An immediate consequence of Lemma 2.8 is that

$$|Y_t| \leq \left(c\mathbb{E}[|\xi|^p + |\alpha|_T^p | \mathcal{F}_t]\right)^{\frac{1}{p}},$$

i.e., Y has an a priori bound which is a continuous supermartingale.

With this estimate we are ready to construct a $\mathbb{L}^p(p > 1)$ solution via inf-(sup-)convolution as in Briand et al [8], [9], [10]. A localization procedure where the a priori bound plays a crucial role is used and the monotone stability result takes the limit.

Theorem 2.9 (Existence) *Let $p > 1$ and (A.1) hold for (F, ξ) . Then there exists a solution to (F, ξ) in $\mathcal{S}^p \times \mathcal{M}^p$.*

Proof. We introduce the notations used throughout the proof. Define the process

$$X_t := \left(c\mathbb{E}[|\xi|^p + |\alpha|_T^p | \mathcal{F}_t]\right)^{\frac{1}{p}},$$

where c is the constant defined in Lemma 2.8. Obviously X is continuous by Itô representation theorem. Moreover, for each $m, n \in \mathbb{N}^+$, set

$$\begin{aligned}\tau_m &:= \inf \{t \geq 0 : |\alpha|_t + X_t \geq m\} \wedge T, \\ \sigma_n &:= \inf \{t \geq 0 : |\alpha|_t \geq n\} \wedge T.\end{aligned}$$

It then follows from the continuity of $|\alpha|$ and X that τ_m and σ_n increase stationarily to T as m, n goes to $+\infty$, respectively. To apply a double approximation procedure we define

$$\begin{aligned}F^{n,k}(t, y, z) &:= \mathbb{I}_{\{t \leq \sigma_n\}} \inf_{y', z'} \{F^+(t, y', z') + n|y - y'| + n|z - z'|\} \\ &\quad - \mathbb{I}_{\{t \leq \sigma_k\}} \inf_{y', z'} \{F^-(t, y', z') + k|y - y'| + k|z - z'|\},\end{aligned}$$

and $\xi^{n,k} := \xi^+ \wedge n - \xi^- \wedge k$.

Before proceeding to the proof we give some useful facts. By Lepeltier and San Martin [23], $F^{n,k}$ is Lipschitz-continuous in (y, z) ; as k goes to $+\infty$, $F^{n,k}$ converges decreasingly uniformly on compact sets to a limit denoted by $F^{n,\infty}$; as n goes to $+\infty$, $F^{n,\infty}$ converges increasingly uniformly on compact sets to F . Moreover, $\|F^{n,k}(\cdot, 0, 0)\|_T$ and $\xi^{n,k}$ are bounded.

Hence, by Briand et al [6], there exists a unique solution $(Y^{n,k}, Z^{n,k}) \in \mathcal{S}^p \times \mathcal{M}^p$ to $(F^{n,k}, \xi^{n,k})$; by comparison theorem, $Y^{n,k}$ is increasing in n and decreasing in k . We are about to take the limit by the monotone stability result.

However, $\|F^{n,k}(\cdot, 0, 0)\|_T$ and $Y^{n,k}$ are not uniformly bounded in general. To overcome this difficulty, we use Lemma 2.8 and work on random time interval where $Y^{n,k}$ and $\|F^{n,k}(\cdot, 0, 0)\|_T$ are uniformly bounded. This is the motivation to introduce X and τ_m . To be more precise, the localization procedure is as follows.

Note that $(F^{n,k}, \xi^{n,k})$ satisfies (A.1) associated with $(\alpha, \beta, \gamma, \varphi, f)$. Hence by Lemma 2.8 (a priori estimate (ii)),

$$\begin{aligned} |Y_t^{n,k}| &\leq \left(c\mathbb{E}[|\xi^{n,k}|^p + |\mathbb{I}_{[0, \sigma_n \vee \sigma_k]} \alpha|_T^p | \mathcal{F}_t] \right)^{\frac{1}{p}} \\ &\leq X_t. \end{aligned} \quad (2.27)$$

In view of the definition of τ_m , we deduce that

$$|Y_{t \wedge \tau_m}^{n,k}| \leq X_{t \wedge \tau_m} \leq m. \quad (2.28)$$

Hence $Y^{n,k}$ is uniformly bounded on $[0, \tau_m]$. Secondly, given $(Y^{n,k}, Z^{n,k})$ which solves $(F^{n,k}, \xi^{n,k})$, it is immediate that $(Y_{\cdot \wedge \tau_m}^{n,k}, \mathbb{I}_{[0, \tau_m]} Z^{n,k})$ solves $(\mathbb{I}_{[0, \tau_m]} F^{n,k}, Y_{\tau_m}^{n,k})$. To make the monotone stability result adaptable, we use a truncation procedure. Define

$$\rho(y) := -\mathbb{I}_{\{y < -m\}}m + \mathbb{I}_{\{|y| \leq m\}}y + \mathbb{I}_{\{y > m\}}m.$$

Hence from (2.28) $(Y_{\cdot \wedge \tau_m}^{n,k}, \mathbb{I}_{[0, \tau_m]} Z^{n,k})$ meanwhile solves $(\mathbb{I}_{[0, \tau_m]}(t) F^{n,k}(t, \rho(y), z), Y_{\tau_m}^{n,k})$. Secondly, we have

$$\begin{aligned} |\mathbb{I}_{[0, \tau_m]}(t) F^{n,k}(t, \rho(y), z)| &\leq \mathbb{I}_{\{t \leq \tau_m\}} (\alpha_t + \varphi(|\rho(y)|) + \gamma|z| + f(|\rho(y)|)|z|^2) \\ &\leq \mathbb{I}_{\{t \leq \tau_m\}} \left(\alpha_t + \varphi(m) + \gamma|z| + \sup_{|y| \leq m} f(|\rho(y)|)|z|^2 \right) \\ &\leq \mathbb{I}_{\{t \leq \tau_m\}} \left(\alpha_t + \varphi(m) + \frac{\gamma^2}{4} + \left(\sup_{|y| \leq m} f(|\rho(y)|) + 1 \right) |z|^2 \right), \end{aligned}$$

where $\sup_{|y| \leq m} f(|\rho(y)|)$ is bounded for each m due to $f(\cdot) \in \mathcal{I}$. Moreover, the definition of τ_m implies $|\alpha|_{\tau_m} \leq m$. Hence we can use the monotone stability result (Kobylanski [22], Briand and Hu [9] or Theorem 3.6) to obtain $(Y^{m,n,\infty}, Z^{m,n,\infty}) \in \mathcal{S}^\infty \times \mathcal{M}^2$ which

solves $(\mathbb{I}_{[0, \tau_m]}(t)F^{n, \infty}(t, \rho(y), z), \inf_k Y_{\tau_m}^{n, k})$. Moreover, $Y_{\cdot \wedge \tau_m}^{m, n, \infty}$ is the \mathbb{P} -a.s. uniform limit of $Y_{\cdot \wedge \tau_m}^{n, k}$ as k goes to $+\infty$. These arguments hold for any $m, n \in \mathbb{N}^+$.

Due to this convergence result we can pass the comparison property to $Y^{m, n, \infty}$. We use the monotone stability result again to the sequence indexed by n to obtain $(\tilde{Y}^m, \tilde{Z}^m) \in \mathcal{S}^\infty \times \mathcal{M}^2$ which solves $(\mathbb{I}_{[0, \tau_m]}(t)F(t, \rho(y), z), \sup_n \inf_k Y_{\tau_m}^{n, k})$. Likewise, \tilde{Y}^m is the \mathbb{P} -a.s. uniform limit of $Y_{\cdot \wedge \tau_m}^{m, n, \infty}$ as n goes to $+\infty$. Hence we obtain from (2.28) that $|\tilde{Y}_t^m| \leq X_{t \wedge \tau_m} \leq m$. Therefore, $(\tilde{Y}^m, \tilde{Z}^m)$ solves $(\mathbb{I}_{[0, \tau_m]}F, \sup_n \inf_k Y_{\tau_m}^{n, k})$, i.e.,

$$\tilde{Y}_{t \wedge \tau_m}^m = \sup_n \inf_k Y_{\tau_m}^{n, k} + \int_{t \wedge \tau_m}^{\tau_m} F(s, \tilde{Y}_s^m, \tilde{Z}_s^m) ds - \int_{t \wedge \tau_m}^{\tau_m} \tilde{Z}_s^m dW_s. \quad (2.29)$$

We recall that the monotone stability result also implies that \tilde{Z}^m is the \mathcal{M}^2 -limit of $\mathbb{I}_{[0, \tau_m]}Z^{n, k}$ as k, n goes to $+\infty$. This fact and previous convergence results give

$$\begin{aligned} \tilde{Y}_{\cdot \wedge \tau_m}^{m+1} &= \tilde{Y}_{\cdot \wedge \tau_m}^m \quad \mathbb{P}\text{-a.s.}, \\ \mathbb{I}_{\{t \leq \tau_m\}} \tilde{Z}_t^{m+1} &= \mathbb{I}_{\{t \leq \tau_m\}} \tilde{Z}_t^m dt \otimes d\mathbb{P}\text{-a.e.} \end{aligned} \quad (2.30)$$

Define (Y, Z) on $[0, T]$ by

$$\begin{aligned} Y_t &:= \mathbb{I}_{\{t \leq \tau_1\}} \tilde{Y}_t^1 + \sum_{m \geq 2} \mathbb{I}_{] \tau_{m-1}, \tau_m]} \tilde{Y}_t^m, \\ Z_t &:= \mathbb{I}_{\{t \leq \tau_1\}} \tilde{Z}_t^1 + \sum_{m \geq 2} \mathbb{I}_{] \tau_{m-1}, \tau_m]} \tilde{Z}_t^m. \end{aligned}$$

By (2.30), we have $Y_{\cdot \wedge \tau_m} = \tilde{Y}_{\cdot \wedge \tau_m}^m$ and $\mathbb{I}_{\{t \leq \tau_m\}} Z_t = \mathbb{I}_{\{t \leq \tau_m\}} \tilde{Z}_t^m$. Hence we can rewrite (2.29) as

$$Y_{t \wedge \tau_m} = \sup_n \inf_k Y_{\tau_m}^{n, k} + \int_{t \wedge \tau_m}^{\tau_m} F(s, Y_s, Z_s) ds - \int_{t \wedge \tau_m}^{\tau_m} Z_s dW_s.$$

By sending m to $+\infty$, we deduce that (Y, Z) solves (F, ξ) . Since $(Y^{n, k}, Z^{n, k})$ verifies Lemma 2.8, we use Fatou's lemma to prove that $(Y, Z) \in \mathcal{S}^p \times \mathcal{M}^p$.

□

Theorem 2.9 proves the existence of a $\mathbb{L}^p(p > 1)$ solution under (A.1) which to our knowledge the most general assumption. For example, (A.1)(ii) allows one to get rid of monotonicity in y which is required by, e.g., Pardoux [27] and Briand et al [7], [6], [10]. Meanwhile, in contrast to these works, the generator can also be quadratic by setting $f(| \cdot |) \in \mathcal{I}$. Hence Theorem 2.9 provides a unified way to construct solutions to both non-quadratic and quadratic BSDEs via the monotone stability result.

On the other hand, Theorem 2.9 is an extension of Bahlali et al [1] which only studies BSDEs with \mathbb{L}^2 integrability and linear-quadratic growth. However, in contrast

to their work, (A.1) is not sufficient in our setting to ensure the existence of a maximal or minimal solution, since the double approximation procedure makes the comparison between solutions impossible.

However, to prove the existence of a maximal or minimal solution is no way impossible. Since we have X as the a priori bound for solutions, we can convert the question of existence into the question of existence for quadratic BSDEs with double barriers. This problem has been solved by introducing the notion of generalized BSDEs; see Essaky and Hassani [16].

Remark. One may ask that as we use a localization procedure, whether f being bounded or integrable only on compact subsets of \mathbb{R} rather than of class \mathcal{I} suffices to ensure the existence result. It turns out that in data in \mathbb{L}^p is not sufficient for such a generalization, and exponential moments integrability is required. Hence, our existence result shall be seen as merely complementary to the quadratic BSDEs studied by Briand and Hu [8], [9] rather than a complete generalization.

Below is an illustrating example with $f = 1$ which clearly doesn't belong to \mathcal{I} . Similar version can be found in Briand et al [10].

Example 2.10 *There exists a solution (Y, Z) in $\mathcal{S}^2 \times \mathcal{M}^2$ to*

$$Y_t = \xi + \int_t^T |Z_s|^2 ds - \int_t^T Z_s dW_s \quad (2.31)$$

if and only if

$$\mathbb{E}[\exp(2\xi)] < +\infty.$$

Proof. (i). \implies . Let $(Y, Z) \in \mathcal{S}^2 \times \mathcal{M}^2$ be a solution to (2.31). By Itô's formula,

$$\exp(2Y_t) = \exp(2Y_0) + \int_0^t \exp(2Y_s) Z_s dW_s.$$

Now we define $\tau_n := \inf\{t \geq 0 : Y_t \geq n\}$ for each $n \in \mathbb{N}^+$. \mathcal{F}_0 being trivial implies that Y_0 is a constant. Hence $\int_0^{\cdot \wedge \tau_n} \exp(2Y_s) Z_s dW_s$ is a bounded martingale, and

$$\mathbb{E}[\exp(2Y_{T \wedge \tau_n})] = \mathbb{E}[\exp(2Y_0)].$$

By Fatou's lemma we obtain $\mathbb{E}[\exp(2\xi)] < +\infty$.

(ii). \Leftarrow . Assume $\mathbb{E}[\exp(2\xi)] < +\infty$. Thanks to Itô representation theorem, we can define $(\tilde{Y}, \tilde{Z}) \in \mathcal{S} \times \mathcal{M}$ by

$$\tilde{Y}_t := \mathbb{E}[\exp(2\xi) | \mathcal{F}_t] = \tilde{Y}_0 + \int_0^t \tilde{Z}_s dW_s.$$

Set $(Y, Z) := (\frac{1}{2} \ln \tilde{Y}, \frac{\tilde{Z}}{2\tilde{Y}})$. By Itô's formula applied to Y , we easily deduce that (Y, Z) solves (2.31). It thus remains to prove $(Y, Z) \in \mathcal{S}^2 \times \mathcal{M}^2$. Since $x \mapsto \ln(x)$ is concave and increasing, Jensen's inequality yields

$$Y_t = \frac{1}{2} \ln (\mathbb{E}[\exp(2\xi)|\mathcal{F}_t]) \geq \mathbb{E}[\xi|\mathcal{F}_t] \geq 0. \quad (2.32)$$

Hence Y is nonnegative. For each $n \in \mathbb{N}^+$, define $\tau_n := \inf \{t \geq 0: \int_0^t |Z_s|^2 ds \geq n\}$. (Y, Z) being a solution to (2.31) implies that

$$\begin{aligned} \int_0^{T \wedge \tau_n} |Z_s|^2 ds &= Y_0 - Y_{T \wedge \tau_n} + \int_0^{T \wedge \tau_n} Z_s dW_s \\ &\leq Y_0 + \int_0^{T \wedge \tau_n} Z_s dW_s. \end{aligned}$$

Hence (2.32) gives

$$\mathbb{E} \left[\left(\int_0^{T \wedge \tau_n} |Z_s|^2 ds \right)^2 \right] \leq 2Y_0^2 + 2\mathbb{E} \left[\left(\int_0^{T \wedge \tau_n} Z_s dW_s \right)^2 \right]. \quad (2.33)$$

Moreover, by Jensen's inequality applied to the left-hand side of (2.33),

$$\mathbb{E} \left[\int_0^{T \wedge \tau_n} |Z_s|^2 ds \right]^2 \leq 2Y_0^2 + 2\mathbb{E} \left[\int_0^{T \wedge \tau_n} |Z_s|^2 ds \right],$$

Using $2a \leq \frac{a^2}{2} + 2$ to the last term of this inequality gives

$$\mathbb{E} \left[\int_0^{T \wedge \tau_n} |Z_s|^2 ds \right] < 4Y_0^2 + 4.$$

Hence, Fatou's lemma yields $Z \in \mathcal{M}^2$. We then use this result and Fatou's lemma to (2.33) to obtain

$$\mathbb{E} \left[\left(\int_0^T |Z_s|^2 ds \right)^2 \right] < +\infty.$$

Finally we deduce from (2.31) that

$$\begin{aligned} \mathbb{E}[(Y^*)^2] &\leq 3\mathbb{E}[|\xi|^2] + 3\mathbb{E} \left[\left(\int_0^T |Z_s|^2 ds \right)^2 \right] + 3\mathbb{E} \left[\left(\int_0^T Z_s dW_s \right)^* \right]^2 \\ &< +\infty. \end{aligned}$$

Hence $(Y, Z) \in \mathcal{S}^2 \times \mathcal{M}^2$.

□

Let us turn to the uniqueness result. Motivated by Briand and Hu [9] or Da Lio and Ley [11] from the point of view of PDEs, we impose a convexity condition so as to use θ -technique which proves to be convenient to treat quadratic generators. We start from comparison theorem and then move to uniqueness and stability result. To this end, the following assumptions on (F, ξ) are needed.

Assumption (A.2) Let $p > 1$. There exist $\beta_1, \beta_2 \in \mathbb{R}$, $\gamma_1, \gamma_2 \geq 0$, an \mathbb{R}^+ -valued Prog-measurable process α , a continuous nondecreasing function $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $\varphi(0) = 0$, $f(|\cdot|) \in \mathcal{I}$ and $F_1, F_2 : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ which are $\text{Prog} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable such that $F = F_1 + F_2$, $|\xi| + |\alpha|_T \in \mathbb{L}^p$ and \mathbb{P} -a.s.

- (i) for any $t \in [0, T]$, $(y, z) \mapsto F(t, y, z)$ is continuous;
- (ii) $F_1(t, y, z)$ is monotonic in y and Lipschitz-continuous in z , and $F_2(t, y, z)$ is monotonic at $y = 0$ and of linear-quadratic growth in z , i.e., for any $t \in [0, T]$, $y, y' \in \mathbb{R}$, $z, z' \in \mathbb{R}^d$,

$$\begin{aligned} \text{sgn}(y - y')(F_1(t, y, z) - F_1(t, y', z)) &\leq \beta_1 |y - y'|, \\ |F_1(t, y, z) - F_1(t, y, z')| &\leq \gamma_1 |z - z'|, \\ \text{sgn}(y)F_2(t, y, z) &\leq \beta_2 |y| + \gamma_2 |z| + f(|y|)|z|^2; \end{aligned}$$

- (iii) for any $t \in [0, T]$, $(y, z) \mapsto F_2(t, y, z)$ is convex;
- (iv) for any $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$,

$$|F(t, y, z)| \leq \alpha_t + \varphi(|y|) + (\gamma_1 + \gamma_2)|z| + f(|y|)|z|^2.$$

Intuitively, (A.2) specifies an additive structure consisting of two classes of BSDEs. The cases where $F_2 = 0$ coincide with classic existence and uniqueness results (see, e.g., Pardoux [28] or Briand et al [7], [6]). When $F_1 = 0$, the BSDEs include those studied by Bahlali et al [4]. Given convexity as an additional requirement, we can prove an existence and uniqueness result in the presence of both components. This can be seen as a general version of the additive structure discussed in Section 2.4 and a complement to the quadratic BSDEs studied by Bahlali et al [4] and Briand and Hu [9].

We start our proof of comparison theorem by observing that (A.2) implies (A.1). Hence the existence of a $\mathbb{L}^p(p > 1)$ solution is ensured.

Theorem 2.11 (Comparison) *Let $p > 1$, and $(Y, Z), (Y', Z') \in \mathcal{S}^p \times \mathcal{M}$ be solutions to $(F, \xi), (F', \xi')$, respectively. If \mathbb{P} -a.s. for any $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$, $F(t, y, z) \leq F'(t, y, z)$, $\xi \leq \xi'$ and F verifies (A.2), then \mathbb{P} -a.s. $Y \leq Y'$.*

Proof. We introduce the notations used throughout the proof. For any $\theta \in (0, 1)$, define

$$\begin{aligned}\delta F_t &:= F(t, Y'_t, Z'_t) - F'(t, Y'_t, Z'_t), \\ \delta_\theta Y &:= Y - \theta Y', \\ \delta Y &:= Y - Y',\end{aligned}$$

and $\delta_\theta Z, \delta Z$, etc. analogously. θ -technique applied to the generators yields

$$\begin{aligned}& F(t, Y_t, Z_t) - \theta F'(t, Y'_t, Z'_t) \\ &= (F(t, Y_t, Z_t) - \theta F(t, Y'_t, Z'_t)) + \theta (F(t, Y'_t, Z'_t) - F'(t, Y'_t, Z'_t)) \\ &= \theta \delta F_t + (F(t, Y_t, Z_t) - \theta F(t, Y'_t, Z'_t)) \\ &= \theta \delta F_t + (F_1(t, Y_t, Z_t) - \theta F_1(t, Y'_t, Z'_t)) + (F_2(t, Y_t, Z_t) - \theta F_2(t, Y'_t, Z'_t)).\end{aligned}\quad (2.34)$$

By (A.2)(iii),

$$\begin{aligned}F_2(t, Y_t, Z_t) &= F_2(t, \theta Y'_t + (1 - \theta) \frac{\delta_\theta Y_t}{1 - \theta}, \theta Z'_t + (1 - \theta) \frac{\delta_\theta Z_t}{1 - \theta}) \\ &\leq \theta F_2(t, Y'_t, Z'_t) + (1 - \theta) F_2(t, \frac{\delta_\theta Y_t}{1 - \theta}, \frac{\delta_\theta Z_t}{1 - \theta}).\end{aligned}$$

Hence we have

$$F_2(t, Y_t, Z_t) - \theta F_2(t, Y'_t, Z'_t) \leq (1 - \theta) F_2(t, \frac{\delta_\theta Y_t}{1 - \theta}, \frac{\delta_\theta Z_t}{1 - \theta}).\quad (2.35)$$

Let u be the function defined in Section 2.2 associated with a function of class \mathcal{I} to be determined later. Denote $u((\delta_\theta Y_t)^+)$, $u'((\delta_\theta Y_t)^+)$, $u''((\delta_\theta Y_t)^+)$ by u_t , u'_t , u''_t , respectively. It is then known from Section 2.2 that $u_t \geq 0$ and $u'_t > 0$. For any $\tau \in \mathcal{T}$, Tanaka's formula applied to $(\delta_\theta Y)^+$, Itô-Krylov formula applied to $u((\delta_\theta Y_t)^+)$ and Lemma 2.3 give

$$\begin{aligned}& |u_{t \wedge \tau}|^p + \frac{p(p-1)}{2} \int_{t \wedge \tau}^\tau \mathbb{I}_{\{\delta_\theta Y_s > 0\}} |u_s|^{p-2} |u'_s|^2 |\delta_\theta Z_s|^2 ds \\ &\leq |u_\tau|^p + p \int_{t \wedge \tau}^\tau \mathbb{I}_{\{\delta_\theta Y_s > 0\}} |u_s|^{p-1} \underbrace{\left(u'_s (F(s, Y_s, Z_s) - \theta F'(s, Y'_s, Z'_s)) - \frac{1}{2} u''_s |\delta_\theta Z_s|^2 \right)}_{:= \Delta_s} ds \\ &\quad - p \int_{t \wedge \tau}^\tau \mathbb{I}_{\{\delta_\theta Y_s > 0\}} |u_s|^{p-1} u'_s \delta_\theta Z_s dW_s.\end{aligned}\quad (2.36)$$

By (2.34), (2.35), (A.2)(ii) and $\delta F \leq 0$, we deduce that, on $\{\delta_\theta Y_s > 0\}$,

$$\Delta_s \leq u'_s \left(F_1(s, Y_s, Z_s) - \theta F_1(s, Y'_s, Z'_s) + \beta_2 (\delta_\theta Y_s)^+ + \gamma_2 |\delta_\theta Z_s| + \frac{f(\frac{|\delta_\theta Y_s|}{1-\theta})}{1-\theta} |\delta_\theta Z_s|^2 \right) - \frac{1}{2} u''_s |\delta_\theta Z_s|^2.$$

To eliminate the quadratic term, we associate u with $\frac{f(\frac{|\cdot|}{1-\theta})}{1-\theta}$, i.e.,

$$\begin{aligned} u(x) &:= \int_0^x \exp\left(2 \int_0^y \frac{f(\frac{|u|}{1-\theta})}{1-\theta} du\right) dy \\ &= \int_0^x \exp\left(2 \int_0^{\frac{y}{1-\theta}} f(|u|) du\right) dy. \end{aligned}$$

Hence, on $\{\delta_\theta Y_s > 0\}$, the above inequality gives

$$\Delta_s \leq u'_s(F_1(s, Y_s, Z_s) - \theta F_1(s, Y'_s, Z'_s) + \beta_2(\delta_\theta Y_s)^+ + \gamma_2|\delta_\theta Z_s|). \quad (2.37)$$

We are about to send θ to 1, and to this end we give some auxiliary facts. Reset $M := \exp(2 \int_0^\infty f(u) du)$. Obviously $1 \leq M < +\infty$. By dominated convergence, for $x \geq 0$, we have

$$\begin{aligned} \lim_{\theta \rightarrow 1} u(x) &= Mx, \\ \lim_{\theta \rightarrow 1} u'(x) &= M\mathbb{I}_{\{x>0\}} + \mathbb{I}_{\{x=0\}}. \end{aligned} \quad (2.38)$$

Taking (2.37) and (2.38) into account, we come back to (2.36) and send θ to 1. Fatou's lemma used to the ds -integral on the left-hand side of (2.36) and dominated convergence used to the rest integrals give

$$\begin{aligned} &((\delta Y_{t \wedge \tau})^+)^p + \frac{p(p-1)}{2} \int_{t \wedge \tau}^\tau \mathbb{I}_{\{\delta Y_s > 0\}} ((\delta Y_s)^+)^{p-2} |\delta Z_s|^2 ds \\ &\leq ((\delta Y_\tau)^+)^p + p \int_{t \wedge \tau}^\tau \mathbb{I}_{\{\delta Y_s > 0\}} ((\delta Y_s)^+)^{p-1} (F_1(s, Y_s, Z_s) - F_1(s, Y'_s, Z'_s) + \beta_2(\delta Y_s)^+ + \gamma_2|\delta Z_s|) ds \\ &\quad - p \int_{t \wedge \tau}^\tau \mathbb{I}_{\{\delta Y_s > 0\}} ((\delta Y_s)^+)^{p-1} \delta Z_s dW_s. \end{aligned} \quad (2.39)$$

Moreover, (A.2)(ii) implies

$$\mathbb{I}_{\{\delta Y_s > 0\}} (F_1(s, Y_s, Z_s) - F_1(s, Y'_s, Z'_s)) \leq \mathbb{I}_{\{\delta Y_s > 0\}} (\beta_1(\delta Y_s)^+ + \gamma_1|\delta Z_s|).$$

We then use this inequality to (2.39). To eliminate the local martingale, we replace τ by a localization sequence $\{\tau_n\}_{n \in \mathbb{N}^+}$ and use the same estimation as in Lemma 2.8 (a priori estimate (ii)).

$$((\delta Y_{t \wedge \tau_n})^+)^p \leq c \mathbb{E}[(\delta Y_{\tau_n})^+)^p | \mathcal{F}_t],$$

where c is a constant only depending on $T, \beta_1, \beta_2, \gamma_1, \gamma_2, p$. Since $Y, Y' \in \mathcal{S}^p$ and \mathbb{P} -a.s. $\xi \leq \xi'$, dominated convergence yields \mathbb{P} -a.s. $Y_t \leq Y'_t$. Finally by the continuity of Y and Y' we conclude that \mathbb{P} -a.s. $Y \leq Y'$.

□

As a byproduct, we obtain the following existence and uniqueness result.

Corollary 2.12 (Uniqueness) *Let (A.2) hold for (F, ξ) . Then there exists a unique solution in $\mathcal{S}^p \times \mathcal{M}^p$.*

Proof. (A.2) implies (A.1). Hence existence result holds. The uniqueness is immediate from Theorem 2.11 (comparison theorem). □

It turns out that a stability result also holds given the convexity condition. We denote (F, ξ) satisfying (A.2) by (F, F_1, F_2, ξ) . We set $\mathbb{N}^0 := \mathbb{N}^+ \cup \{0\}$.

Proposition 2.13 (Stability) *Let $p > 1$. Let $(F^n, F_1^n, F_2^n, \xi^n)_{n \in \mathbb{N}^0}$ satisfy (A.2) associated with $(\alpha^n, \beta_1, \beta_2, \gamma_1, \gamma_2, \varphi, f)$, and (Y^n, Z^n) be their unique solutions in $\mathcal{S}^p \times \mathcal{M}^p$, respectively. If $\xi^n - \xi \rightarrow 0$ and $\int_0^T |F^n - F^0|(s, Y_s^0, Z_s^0) ds \rightarrow 0$ in \mathbb{L}^p as n goes to $+\infty$, then (Y^n, Z^n) converges to (Y, Z) in $\mathcal{S}^p \times \mathcal{M}^p$.*

Proof. We prove the stability result in the spirit of Theorem 2.11 (comparison theorem). For any $\theta \in (0, 1)$, define

$$\begin{aligned}\delta F_t^n &:= F^0(t, Y_t^0, Z_t^0) - F^n(t, Y_t^n, Z_t^n), \\ \delta_\theta Y^n &:= Y^0 - \theta Y^n, \\ \delta Y^n &:= Y^0 - Y^n,\end{aligned}$$

and $\delta_\theta Z^n, \delta Z^n$, etc. analogously. We observe the θ -difference of the generators. Likewise, (A.2)(iii) implies that

$$\begin{aligned}& F^0(t, Y_t^0, Z_t^0) - \theta F^n(t, Y_t^n, Z_t^n) \\ &= \delta F_t^n + (F^n(t, Y_t^0, Z_t^0) - \theta F^n(t, Y_t^n, Z_t^n)) \\ &\leq \delta F_t^n + (F_1^n(t, Y_t^0, Z_t^0) - \theta F_1^n(t, Y_t^n, Z_t^n)) + (1 - \theta) F_2^n(t, \frac{\delta_\theta Y_s^n}{1 - \theta}, \frac{\delta_\theta Z_s^n}{1 - \theta}).\end{aligned}$$

We first prove convergence of Y^n and later use it to show that Z^n also converges.

(i). By exactly the same arguments as in Theorem 2.11 but keeping δF_t^n along the deductions, we obtain

$$\begin{aligned}& ((\delta Y_t^n)^+)^p + \frac{p(p-1)}{2} \int_t^T \mathbb{I}_{\{\delta Y_s^n > 0\}} ((\delta Y_s^n)^+)^{p-2} |\delta Z_s^n|^2 ds \\ &\leq ((\delta \xi^n)^+)^p + p \int_t^T \mathbb{I}_{\{\delta Y_s^n > 0\}} ((\delta Y_s^n)^+)^{p-1} (|\delta F_s^n| + (\beta_1 + \beta_2)(\delta Y_s^n)^+ + (\gamma_1 + \gamma_2)|\delta Z_s^n|) ds \\ &- p \int_t^T \mathbb{I}_{\{\delta Y_s^n > 0\}} ((\delta Y_s^n)^+)^{p-1} \delta Z_s^n dW_s,\end{aligned}\tag{2.40}$$

By the same way of estimation as in Lemma 2.8 (a priori estimate (ii)), we obtain

$$\mathbb{E}[\left(((\delta Y^n)^+)^*\right)^p] \leq c \left(\mathbb{E}[\left((\delta \xi^n)^+\right)^p] + \mathbb{E}[|\delta F^n|_T^p] \right),$$

where c is a constant only depending on $T, \beta_1, \beta_2, \gamma_1, \gamma_2, p$. Interchanging Y^0 and Y^n and analogous deductions then yield

$$\mathbb{E}[\left(((-\delta Y^n)^+)^*\right)^p] \leq c \left(\mathbb{E}[\left((-\delta \xi^n)^+\right)^p] + \mathbb{E}[|\delta F^n|_T^p] \right).$$

Hence a combination of the two inequalities implies the convergence of Y^n .

(ii). To prove the convergence of Z^n , we combine the arguments in Lemma 2.7 (a priori estimate (i)) and Theorem 2.11. To this end, we introduce the function v defined in Section 2.2 associated with a function of class \mathcal{I} to be determined later. By Itô-Krylov formula,

$$\begin{aligned} v(\delta_\theta Y_0^n) &= v(\delta_\theta \xi^n) + \int_0^T v'(\delta_\theta Y_s^n) (F^0(s, Y_s^0, Z_s^0) - \theta F^n(s, Y_s^n, Z_s^n)) ds \\ &\quad - \frac{1}{2} \int_0^T v''(\delta_\theta Y_s^n) |\delta_\theta Z_s^n|^2 ds - \int_0^T v'(\delta_\theta Y_s^n) \delta_\theta Z_s^n dW_s. \end{aligned} \quad (2.41)$$

Note that (A.2)(ii)(iii) and $v'(\delta_\theta Y_s^n) = \text{sgn}(\delta_\theta Y_s^n) |v'(\delta_\theta Y_s^n)|$ give

$$\begin{aligned} v'(\delta_\theta Y_s^n) (F^0(s, Y_s^0, Z_s^0) - \theta F^n(s, Y_s^n, Z_s^n)) &\leq |v'(\delta_\theta Y_s^n)| |\delta F_s^n| \\ &\quad + |v'(\delta_\theta Y_s^n)| \text{sgn}(\delta_\theta Y_s^n) (F_1^n(s, Y_s^0, Z_s^0) - \theta F_1^n(s, Y_s^n, Z_s^n)) \\ &\quad + |v'(\delta_\theta Y_s^n)| \left(\beta_2 |\delta_\theta Y_s^n| + \gamma_2 |\delta_\theta Z_s^n| + \frac{f(\frac{|\delta_\theta Y_s^n|}{1-\theta})}{1-\theta} |\delta_\theta Z_s^n|^2 \right). \end{aligned} \quad (2.42)$$

We associate v with $\frac{f(\frac{|\cdot|}{1-\theta})}{1-\theta}$ so as to eliminate the quadratic term. Note that

$$\begin{aligned} \lim_{\theta \rightarrow 1} v(x) &= \frac{1}{2} |x|^2, \\ \lim_{\theta \rightarrow 1} v'(x) &= x. \end{aligned} \quad (2.43)$$

With (2.42), (2.43) and (A.2)(ii), we come back to (2.41) and send θ to 1. This gives

$$\begin{aligned} \frac{1}{2} \int_0^T |\delta Z_s^n|^2 ds &\leq \frac{1}{2} |\delta \xi^n|^2 + \int_0^T |\delta Y_s^n| (|\delta F_s^n| + (|\beta_1| + |\beta_2|) |\delta Y_s^n| + (\gamma_1 + \gamma_2) |\delta Z_s^n|) ds \\ &\quad - \int_0^T \delta Y_s^n \delta Z_s^n dW_s. \end{aligned}$$

Now we use the same way of estimation as in Lemma 2.7 to obtain

$$\mathbb{E} \left[\left(\int_0^T |\delta Z_s^n|^2 ds \right)^{\frac{p}{2}} \right] \leq c \mathbb{E} [((\delta Y^n)^*)^p + \|\delta F^n\|_T^p],$$

where c is a constant only depending on $T, \beta_1, \beta_2, \gamma_1, \gamma_2, p$. The convergence of Z^n is then immediate from (i). □

Remark. So far we have obtained the existence and uniqueness of a $\mathbb{L}^p(p > 1)$ solution. The solvability for $p = 1$ is not included due to the failure of Lemma 2.8 (a priori estimate (ii)). One may overcome this difficulty by imposing additional structure conditions as in Briand et al [6], [8]. To save pages the analysis of \mathbb{L}^1 solutions is hence omitted.

2.6 Applications to Quadratic PDEs

In this section, we give an application of our results to quadratic PDEs. More precisely, we prove the probabilistic representation for the nonlinear Feymann-Kac formula associated with the BSDEs in our study. Let us consider the following semilinear PDE

$$\begin{aligned} \partial_t u(t, x) + \mathcal{L}u(t, x) + F(t, x, u(t, x), \sigma^\top \nabla_x u(t, x)) &= 0, \\ u(T, \cdot) &= g, \end{aligned} \tag{2.44}$$

where \mathcal{L} is the infinitesimal generator of the solution X^{t_0, x_0} to the Markovian SDE

$$X_t = x_0 + \int_{t_0}^t b(s, X_s) ds + \int_{t_0}^t \sigma(s, X_s) dB_s, \tag{2.45}$$

for any $(t_0, x_0) \in [0, T] \times \mathbb{R}^n$, $t \in [t_0, T]$. Denote a solution to the BSDE

$$Y_t = g(X_T^{t_0, x_0}) + \int_t^T F(s, X_s^{t_0, x_0}, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \quad t \in [t_0, T], \tag{2.46}$$

by $(Y^{t_0, x_0}, Z^{t_0, x_0})$ or (Y, Z) when there is no ambiguity. The probabilistic representation for nonlinear Feymann-Kac formula consists of proving that, in Markovian setting, $u(t, x) := Y_t^{t, x}$ is a solution at least in the viscosity sense to (2.44) when the source of nonlinearity F is quadratic in $\nabla_x u(t, x)$ and g is an unbounded function. To put it more precisely, let us introduce the FBSDEs.

The Forward Markovian SDEs. Let $b : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $\sigma : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{n \times d}$ be continuous functions and assume there exists $\beta \geq 0$ such that \mathbb{P} -a.s. for any $t \in [0, T]$,

$|b(t, 0)| + |\sigma(t, 0)| \leq \beta$ and $b(t, x), \sigma(t, x)$ are Lipschitz-continuous in x , i.e., \mathbb{P} -a.s. for any $t \in [0, T]$, $x, x' \in \mathbb{R}^n$,

$$|b(t, x) - b(t, x')| + |\sigma(t, x) - \sigma(t, x')| \leq \beta|x - x'|.$$

Then for any $(t_0, x_0) \in [0, T] \times \mathbb{R}^n$, (2.45) has a unique solution X^{t_0, x_0} in \mathcal{S}^p for any $p \geq 1$.

The Markovian BSDE. We continue with the setting of the forward equations above. Set $q \geq 1$. Let $F_1, F_2 : [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$, $g : \mathbb{R}^n \rightarrow \mathbb{R}$ be continuous functions, $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ a continuous nondecreasing function with $\varphi(0) = 0$ and $f(|\cdot|) \in \mathcal{I}$, and assume moreover $F = F_1 + F_2$ such that

- (i) $F_1(t, x, y, z)$ is monotonic in y and Lipschitz-continuous in z , and $F_2(t, x, y, z)$ is monotonic at $y = 0$ and of linear-quadratic growth in z , i.e., for any $(t, x) \in [0, T] \times \mathbb{R}^n$, $y, y' \in \mathbb{R}$, $z, z' \in \mathbb{R}^d$,

$$\begin{aligned} \operatorname{sgn}(y - y')(F_1(t, x, y, z) - F_1(t, x, y', z)) &\leq \beta|y - y'|, \\ |F_1(t, x, y, z) - F_1(t, x, y, z')| &\leq \beta|z - z'|, \\ \operatorname{sgn}(y)F_2(t, x, y, z) &\leq \beta|y| + \beta|z| + f(|y|)|z|^2; \end{aligned}$$

- (ii) $(y, z) \mapsto F_2(t, x, y, z)$ is convex ;

- (iii) for any $(t, x, y, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d$,

$$\begin{aligned} |F(t, x, y, z)| &\leq \beta(1 + |x|^q + 2|z|) + \varphi(|y|) + f(|y|)|z|^2, \\ |g(x)| &\leq \beta(1 + |x|^q). \end{aligned}$$

Since $X^{t_0, x_0} \in \mathcal{S}^p$ for any $p \geq 1$, the above structure conditions on F and g allow one to use Corollary 2.12 to construct a unique solution $(Y^{t_0, x_0}, Z^{t_0, x_0})$ in $\mathcal{S}^p \times \mathcal{M}^p$ to (2.46) for any $p > 1$. Moreover, by standard arguments, $Y_{t_0}^{t_0, x_0}$ is deterministic for any $(t_0, x_0) \in [0, T] \times \mathbb{R}^n$. Hence $u(t, x)$ defined as $Y_t^{t, x}$ is a deterministic function. With this fact we now turn to the main result of this section: u is a viscosity solution to (2.44). Before our proof let us recall the definition of a viscosity solution.

Viscosity Solution. A continuous function $u : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ is called a viscosity subsolution (respectively supersolution) to (2.44) if $u(T, x) \leq g(x)$ (respectively $u(T, x) \geq g(x)$) and for any smooth function ϕ such that $u - \phi$ reaches the local maximum (respectively local minimum) at (t_0, x_0) , we have

$$\partial_t \phi(t_0, x_0) + \mathcal{L}\phi(t_0, x_0) + F(t_0, x_0, u(t_0, x_0), \sigma^\top \nabla_x \phi(t_0, x_0)) \geq 0 \text{ (respectively } \leq 0).$$

A function u is called a viscosity solution to (2.44) if it is both a viscosity subsolution and supersolution.

Proposition 2.14 *Given the above assumptions, $u(t, x)$ is continuous with*

$$|u(t, x)| \leq c(1 + |x|^q),$$

where c is a constant. Moreover, u is a viscosity solution to (2.44).

Proof. Due to the Lipschitz-continuity of b and σ , $X^{t,x}$ is continuous in (t, x) , e.g., in mean square sense. The continuity of u is then an immediate consequence of Theorem 2.13 (stability). The proof relies on standard arguments and hence is omitted. By Lemma 2.8 (a priori estimate (ii)), we prove that u satisfies the above polynomial growth. It thus remains to prove that u is a viscosity solution to (2.44).

Let ϕ be a smooth function such that $u - \phi$ reaches local maximum at (t_0, x_0) . Without loss of generality we assume that the local maximum is global and $u(t_0, x_0) = \phi(t_0, x_0)$. We aim at proving

$$\partial_t \phi(t_0, x_0) + \mathcal{L}\phi(t_0, x_0) + F(t_0, x_0, u(t_0, x_0), \sigma^\top \nabla_x \phi(t_0, x_0)) \geq 0.$$

From (2.46) we obtain

$$Y_t = Y_{t_0} - \int_{t_0}^t F(s, X_s^{t_0, x_0}, Y_s, Z_s) ds + \int_{t_0}^t Z_s dW_s.$$

By Itô's formula,

$$\phi(t, X_t^{t_0, x_0}) = \phi(t_0, x_0) + \int_{t_0}^t \{ \partial_s \phi + \mathcal{L}\phi \}(s, X_s^{t_0, x_0}) ds + \int_{t_0}^t \sigma^\top \nabla_x \phi(s, X_s^{t_0, x_0}) dW_s.$$

Now we take any $t \in [t_0, T]$. Note that the existence of a unique solution to (2.45) and (2.46) implies by Markov property that $Y_t = u(t, X_t^{t_0, x_0})$. Hence, $\phi(t, X_t^{t_0, x_0}) \geq u(t, X_t^{t_0, x_0}) = Y_t$. By touching property, on the set $\{\phi(t, X_t^{t_0, x_0}) = Y_t\}$ we have

$$\begin{aligned} \partial_t \phi(t, X_t^{t_0, x_0}) + \mathcal{L}\phi(t, X_t^{t_0, x_0}) + F(t, X_t^{t_0, x_0}, Y_t, Z_t) &\geq 0 \quad \mathbb{P}\text{-a.s.}, \\ \sigma^\top \nabla_x \phi(t, X_t^{t_0, x_0}) - Z_t &= 0 \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

Now we set $t = t_0$. We have $\phi(t_0, X_{t_0}^{t_0, x_0}) = \phi(t_0, x_0) = u(t_0, x_0) = Y_{t_0}$. Moreover, the above equality implies $Z_{t_0} = \sigma^\top \nabla_x \phi(t_0, x_0)$. Plugging the two equalities into the above inequality gives

$$\partial_t \phi(t_0, x_0) + \mathcal{L}\phi(t_0, x_0) + F(t_0, x_0, u(t_0, x_0), \sigma^\top \nabla_x \phi(t_0, x_0)) \geq 0.$$

Hence u is a viscosity subsolution to (2.44). u being a viscosity supersolution and thus a viscosity solution can be proved analogously. □

Chapter 3

Quadratic Semimartingale BSDEs

3.1 Preliminaries

The objectives of our study in this chapter are quadratic BSDEs driven by continuous local martingales. We fix the time horizon $T > 0$, and work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ satisfying the usual conditions of right-continuity and \mathbb{P} -completeness. \mathcal{F}_0 is the \mathbb{P} -completion of the trivial σ -algebra. Any measurability will refer to the filtration $(\mathcal{F}_t)_{t \in [0, T]}$. In particular, Prog denotes the progressive σ -algebra on $\Omega \times [0, T]$. We assume the filtration is *continuous*, in the sense that all local martingales have \mathbb{P} -a.s. continuous sample paths. $M = (M^1, \dots, M^d)^\top$ stands for a fixed d -dimensional continuous local martingale. By *continuous semimartingale setting* we mean: M doesn't have to be a Brownian motion; the filtration is not necessarily generated by M which is usually seen as the main source of randomness. Hence in various concrete situations there may be a continuous local martingale N strongly orthogonal to M . As mentioned in the introduction, we exclusively study \mathbb{R} -valued BSDEs. They can be written as

$$Y_t = \xi + \int_t^T (\mathbf{1}^\top d\langle M \rangle_s F(s, Y_s, Z_s) + g_s d\langle N \rangle_s) - \int_t^T (Z_s dM_s + dN_s),$$

where $\mathbf{1} := (1, \dots, 1)^\top$, ξ is an \mathbb{R} -valued \mathcal{F}_T -measurable random variable, $F : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a $\text{Prog} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable random function and g is an \mathbb{R} -valued Prog -measurable bounded process. $\int_0^\cdot (Z_s dM_s + dN_s)$, sometimes denoted by $Z \cdot M + N$, refers to the vector stochastic integral; see Shiryaev and Cherny [30]. The equations defined in this way encode the matrix-valued process $\langle M \rangle$ which is not amenable to analysis. Therefore we rewrite the BSDEs by factorizing $\langle M \rangle$. This procedure separates the matrix property from its nature as a measure. It can also be regarded as a reduction of dimensionality.

There are many ways to factorize $\langle M \rangle$; see, e.g., Section III. 4a, Jacod and Shiryaev [20]. We can and choose $A := \arctan \left(\sum_{i=1}^d \langle M^i \rangle \right)$. By Kunita-Watanabe inequality, we

deduce the absolute continuity of $\langle M^i, M^j \rangle$ with respect to A . Note that such choice makes A continuous, increasing and bounded. Moreover, by Radon-Nikodým theorem and Cholesky decomposition, there exists a matrix-valued Prog-measurable process λ such that $\langle M \rangle = (\lambda^\top \lambda) \cdot A$. As will be seen later, our results don't rely on the specific choice of A but only on its boundedness. In particular, if M is a d -dimensional Brownian motion, we may choose $A_t = t$ and λ to be the identity matrix.

The second advantage of factorizing $\langle M \rangle$ is that

$$\mathbf{1}^\top d\langle M \rangle_s F(s, Y_s, Z_s) = \mathbf{1}^\top \lambda_s^\top \lambda_s F(s, Y_s, Z_s) dA_s,$$

where $f(t, y, z) := \mathbf{1}^\top \lambda_s^\top \lambda_s F(s, y, z)$ is \mathbb{R} -valued. Such reduction of dimensionality makes it easier to formulate the difference of two equations as frequently appears in comparison theorem and uniqueness. Hence, we may reformulate the BSDEs as follows.

BSDEs: Definition and Solutions. Let A be an \mathbb{R} -valued continuous nondecreasing bounded adapted process such that $\langle M \rangle = (\lambda^\top \lambda) \cdot A$ for some matrix-valued Prog-measurable process λ , $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ a $\text{Prog} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable random function, g an \mathbb{R} -valued Prog-measurable bounded process and ξ an \mathbb{R} -valued \mathcal{F}_T -measurable random variable. The semimartingale BSDEs are written as

$$Y_t = \xi + \int_t^T (f(s, Y_s, Z_s) dA_s + g_s d\langle N \rangle_s) - \int_t^T (Z_s dM_s + dN_s). \quad (3.1)$$

We call a process (Y, Z, N) or $(Y, Z \cdot M + N)$ a *solution* to (3.1), if Y is an \mathbb{R} -valued continuous adapted process, Z is an \mathbb{R}^d -valued Prog-measurable process and N is an \mathbb{R} -valued continuous local martingale strongly orthogonal to M , such that \mathbb{P} -a.s. $\int_0^T Z_s^\top d\langle M \rangle_s Z_s < +\infty$ and $\int_0^T |f(s, Y_s, Z_s)| dA_s < +\infty$, and (3.1) holds \mathbb{P} -a.s. for all $t \in [0, T]$.

Note that the factorization of $\langle M \rangle$ gives $\int_0^T Z_s^\top d\langle M \rangle_s Z_s = \int_0^T |\lambda_s Z_s|^2 dA_s$. Hence we don't distinguish these two integrals in all situations. $\int_0^T Z_s^\top d\langle M \rangle_s Z_s < +\infty$ \mathbb{P} -a.s. ensures that Z is integrable with respect to M in the sense of vector stochastic integration. As a result, $Z \cdot M$ is a continuous local martingale. M and N being continuous and strongly orthogonal implies that $\langle M^i, N \rangle = 0$ for $i = 1, \dots, d$. We call f the *generator*, ξ the *terminal value* and $(\xi, \int_0^T |f(s, 0, 0)| dA_s)$ the *data*. In our study, the integrability property of the data determines the estimates for a solution. The conditions imposed on the generator are called the *structure conditions*. For notational convenience, we sometimes write (f, g, ξ) instead of (3.1) to denote the above BSDE. Finally, (3.1) is called *quadratic* if f has at most quadratic growth in z or g is not indistinguishable from 0.

Regarding the existence results, most literature requires g to be a constant; see, e.g., [15], [26], [25]. The reason is that $g \cdot \langle N \rangle$ can be eliminated via exponential transform only if g is a constant. Tevzadze [31] allows g to be any bounded process but their results are less general in several aspects. We also point out that in mathematical finance, g usually appears as a constant; see, e.g., [24], [5], [19], [17].

We take a further step by studying bounded and unbounded solutions to BSDEs associated with any bounded process g , and with monotonicity at $y = 0$ and at most

quadratic growth in z . The conditions to our knowledge are the most general compared to existing literature. We start from bounded solutions to Lipschitz-quadratic BSDEs (see Section 3.2) and then extend the results to general quadratic BSDEs (see Section 3.3, 3.4).

Let us close this section by introducing all required notations for this chapter. \ll stands for the strong order of nondecreasing processes, stating that the difference is nondecreasing. For any random variable or process Y , we say Y has some property if this is true except on a \mathbb{P} -null subset of Ω . Hence we omit “ \mathbb{P} -a.s.” in situations without ambiguity. Define $\text{sgn}(x) = \mathbb{I}_{\{x \neq 0\}} \frac{x}{|x|}$. For any random variable X , define $\|X\|_\infty$ to be its essential supremum. For any càdlàg adapted process Y , set $Y_{s,t} := Y_t - Y_s$ and $Y^* := \sup_{t \in [0,T]} |Y_t|$. For any Prog-measurable process H , set $|H|_{s,t} := \int_s^t H_u dA_u$ and $|H|_t := |H|_{0,t}$. \mathcal{T} stands for the set of all stopping times valued in $[0, T]$ and \mathcal{S} denotes the space of continuous adapted processes. For later use we specify the following spaces under \mathbb{P} .

- \mathcal{S}^∞ : the space of bounded processes $Y \in \mathcal{S}$ with $\|Y\| := \|Y^*\|_\infty$; \mathcal{S}^∞ is a Banach space;
- \mathcal{M} : the set of continuous local martingales starting from 0; for any \mathbb{R}^d -valued Prog-measurable process Z with $\int_0^T Z_s^\top d\langle M \rangle_s Z_s < +\infty$, $Z \cdot M \in \mathcal{M}$;
- $\mathcal{M}^p(p \geq 1)$: the set of $\widetilde{M} \in \mathcal{M}$ with

$$\|\widetilde{M}\|_{\mathcal{M}^p} := \left(\mathbb{E} \left[\langle \widetilde{M} \rangle_T^{\frac{p}{2}} \right] \right)^{\frac{1}{p}} < +\infty;$$

in particular, \mathcal{M}^2 is a Hilbert space;

- \mathcal{M}^{BMO} : the set of BMO martingales $\widetilde{M} \in \mathcal{M}$ with

$$\|\widetilde{M}\|_{BMO} := \sup_{\tau \in \mathcal{T}} \left\| \mathbb{E} \left[\langle \widetilde{M} \rangle_{\tau, T} \middle| \mathcal{F}_\tau \right]^{\frac{1}{2}} \right\|_\infty;$$

\mathcal{M}^{BMO} is a Banach space.

\mathcal{M}^2 being a Hilbert space is crucial to proving convergence of the martingale parts in the monotone stability result of quadratic BSDEs (see, e.g., Kobylanski [22], Briand and Hu [9], Morlais [26] or Section 3.3). Other spaces are also Banach under suitable norms; we will not present these facts in more detail since they are not involved in our study.

Finally, for any local martingale \widetilde{M} , we call $\{\sigma_n\}_{n \in \mathbb{N}^+} \subset \mathcal{T}$ a *localizing sequence* if σ_n increases stationarily to T as n goes to $+\infty$ and $\widetilde{M}_{\cdot \wedge \sigma_n}$ is a martingale for any $n \in \mathbb{N}^+$.

3.2 Bounded Solutions to Lipschitz-quadratic BSDEs

This section takes one step in solving quadratic BSDEs and consists in the study of equations with Lipschitz-continuous generators. In contrast to El Karoui and Huang [15], we allow the presence of $g \cdot \langle N \rangle$. We point out that similar results for linear-quadratic generators have been studied by Tevzadze [31], but the case of Lipschitz-continuity is not available in that work. Due to its importance for regularizations of quadratic BSDEs, we study existence and uniqueness results for equations of this particular type in the first step. To this end, we assume

Assumption (A.1) There exist $\beta, \gamma \geq 0$ such that $\|\xi\|_\infty + \| |f(\cdot, 0, 0)| |_T \|_\infty < +\infty$ and f is Lipschitz-continuous in (y, z) , i.e., \mathbb{P} -a.s. for any $t \in [0, T]$, $y, y' \in \mathbb{R}$, $z, z' \in \mathbb{R}^d$,

$$|f(t, y, z) - f(t, y', z')| \leq \beta|y - y'| + \gamma|\lambda_t(z - z')|.$$

Due to the presence of $g \cdot \langle N \rangle$, we call the BSDE (f, g, ξ) satisfying (A.1) *Lipschitz-quadratic*. Given (A.1), we are about to construct a solution in the space $\mathcal{B} := \mathcal{S}^\infty \times \mathcal{M}^{BMO}$ equipped with the norm

$$\|(Y, Z \cdot M + N)\| := (\|Y\|^2 + \|Z \cdot M + N\|_{BMO}^2)^{\frac{1}{2}},$$

for $(Y, Z \cdot M + N) \in \mathcal{S}^\infty \times \mathcal{M}^{BMO}$. Clearly $(\mathcal{B}, \|\cdot\|)$ is Banach. As a preliminary result, we claim that the existence result holds given sufficiently small data.

Theorem 3.1 (Existence (i)) *If (f, g, ξ) satisfies (A.1) with*

$$\|\xi\|_\infty^2 + 8\| |f(\cdot, 0, 0)| |_T \|_\infty^2 \leq \frac{1}{64} \exp\left(-\|A\|(8\beta^2\|A\| + 8\gamma^2)\right) \quad (3.2)$$

and \mathbb{P} -a.s. $|g| \leq \tilde{g} := \frac{1}{8}$, then there exists a solution in $(\mathcal{B}, \|\cdot\|)$.

Proof. To overcome the difficulty arising from the Lipschitz-continuity, we use Banach fixed point theorem under an equivalent norm. Set $\rho \geq 0$ to be determined later. For any $X \in \mathbb{L}^\infty, Y \in \mathcal{S}^\infty$ and $\widetilde{M} \in \mathcal{M}^{BMO}$, set $\|X\|_{\infty, \rho} := \|e^{\frac{\rho}{2}A_T}X\|_\infty$, $\|Y\|_\rho := \|e^{\frac{\rho}{2}A}Y\|$ and $\|\widetilde{M}\|_{BMO, \rho} := \|e^{\frac{\rho}{2}A} \cdot \widetilde{M}\|_{BMO}$; for $(Y, Z \cdot M + N) \in \mathcal{B}$, set

$$\|(Y, Z \cdot M + N)\|_\rho := (\|Y\|_\rho^2 + \|Z \cdot M + N\|_{BMO, \rho}^2)^{\frac{1}{2}}.$$

Since A is bounded, $\|\cdot\|_\rho$ is equivalent to the original norm for each space. Hence $(\mathcal{B}, \|\cdot\|_\rho)$ is also a Banach space. For any $R \geq 0$, define

$$\mathbf{B}_R := \{(Y, Z \cdot M + N) \in \mathcal{B} : \|(Y, Z \cdot M + N)\|_\rho \leq R\}.$$

We show by Banach fixed point theorem that there exists a unique solution in \mathbf{B}_R with $R = \frac{1}{2}$. To this end, we define $\mathbf{F} : (\mathbf{B}_R, \|\cdot\|_\rho) \rightarrow (\mathcal{B}, \|\cdot\|_\rho)$ such that for any $(y, z \cdot M + n) \in \mathbf{B}_R$, $(Y, Z \cdot M + N) := \mathbf{F}((y, z \cdot M + n))$ solves

$$Y_t = \xi + \int_t^T (f(s, y_s, z_s) dA_s + g_s d\langle n \rangle_s) - \int_t^T (Z_s dM_s + dN_s).$$

Indeed, such (Y, Z, N) uniquely exists due to martingale representation theorem. Moreover, by standard estimates, $(Y, Z \cdot M + N) \in (\mathcal{B}, \|\cdot\|_\rho)$.

(i). We show $\mathbf{F}(\mathbf{B}_R) \subset \mathbf{B}_R$. For any $\tau \in \mathcal{T}$, Itô's formula applied to $e^{\rho A} Y^2$ yields

$$\begin{aligned} e^{\rho A_\tau} |Y_\tau|^2 + \rho \mathbb{E} \left[\int_\tau^T e^{\rho A_s} Y_s^2 dA_s \middle| \mathcal{F}_\tau \right] + \mathbb{E} \left[\int_\tau^T e^{\rho A_s} \left(Z_s^\top d\langle M \rangle_s Z_s + d\langle N \rangle_s \right) \middle| \mathcal{F}_\tau \right] \\ \leq \|\xi\|_{\infty, \rho}^2 + 2\mathbb{E} \left[\int_\tau^T e^{\rho A_s} |Y_s| |f(s, y_s, z_s)| dA_s \middle| \mathcal{F}_\tau \right] + 2\mathbb{E} \left[\int_\tau^T e^{\rho A_s} |Y_s| |g_s| d\langle n \rangle_s \middle| \mathcal{F}_\tau \right]. \end{aligned} \quad (3.3)$$

By (A.1),

$$|Y_s| |f(s, y_s, z_s)| \leq |Y_s| |f(s, 0, 0)| + \beta |Y_s| |y_s| + \gamma |Y_s| |\lambda_s z_s|.$$

We plug this inequality into (3.3) and estimate each term on the right-hand side. Using $2ab \leq \frac{1}{8}a^2 + 8b^2$ gives

$$\begin{aligned} 2\mathbb{E} \left[\int_\tau^T e^{\rho A_s} |Y_s| |f(s, 0, 0)| dA_s \middle| \mathcal{F}_\tau \right] &\leq \frac{1}{8} \|Y\|_\rho^2 + 8\mathbb{E} \left[\int_\tau^T e^{\frac{\rho}{2} A_s} |f(s, 0, 0)| dA_s \middle| \mathcal{F}_\tau \right]^2 \\ &\leq \frac{1}{8} \|Y\|_\rho^2 + 8 \| |f(\cdot, 0, 0)| \|_T^2_{\infty, \rho}, \end{aligned}$$

$$\begin{aligned} 2\beta \mathbb{E} \left[\int_\tau^T e^{\rho A_s} |Y_s| |y_s| dA_s \middle| \mathcal{F}_\tau \right] &\leq \frac{1}{8} \|y\|_\rho^2 + 8\beta^2 \mathbb{E} \left[\int_\tau^T e^{\frac{\rho}{2} A_s} |Y_s| dA_s \middle| \mathcal{F}_\tau \right]^2 \\ &\leq \frac{1}{8} \|y\|_\rho^2 + 8\beta^2 \|A\| \mathbb{E} \left[\int_\tau^T e^{\rho A_s} |Y_s|^2 dA_s \middle| \mathcal{F}_\tau \right], \end{aligned}$$

$$2\gamma \mathbb{E} \left[\int_\tau^T e^{\rho A_s} |Y_s| |\lambda_s z_s| dA_s \middle| \mathcal{F}_\tau \right] \leq \frac{1}{8} \|z \cdot M\|_{BMO, \rho}^2 + 8\gamma^2 \mathbb{E} \left[\int_\tau^T e^{\rho A_s} |Y_s|^2 dA_s \middle| \mathcal{F}_\tau \right],$$

$$\begin{aligned} 2\mathbb{E} \left[\int_\tau^T e^{\rho A_s} |Y_s| |g_s| d\langle N \rangle_s \middle| \mathcal{F}_\tau \right] &\leq \frac{1}{8} \|Y\|_\rho^2 + 8\tilde{g}^2 \mathbb{E} \left[\int_\tau^T e^{\frac{\rho}{2} A_s} d\langle N \rangle_s \middle| \mathcal{F}_\tau \right]^2 \\ &\leq \frac{1}{8} \|Y\|_\rho^2 + 8\tilde{g}^2 \|n\|_{BMO, \rho}^4. \end{aligned}$$

Set $\rho := 8\beta^2\|A\| + 8\gamma^2$ so as to eliminate $\mathbb{E}\left[\int_\tau^T e^{\rho A_s} Y_s^2 dA_s \middle| \mathcal{F}_\tau\right]$ on both sides. Hence (3.3) gives

$$\begin{aligned} e^{\rho A_\tau} |Y_\tau|^2 + \mathbb{E}\left[\int_\tau^T e^{\rho A_s} \left(Z_s^\top d\langle M \rangle_s Z_s + d\langle N \rangle_s\right) \middle| \mathcal{F}_\tau\right] \\ \leq \|\xi\|_{\infty, \rho}^2 + 8\|f(\cdot, 0, 0)\|_T^2 + \frac{1}{4}\|Y\|_\rho^2 \\ + \frac{1}{8}(\|y\|_\rho^2 + \|z \cdot M\|_{BMO, \rho}^2) + 8\tilde{g}^2 \|n\|_{BMO, \rho}^4. \end{aligned} \quad (3.4)$$

Taking essential supremum and supremum over all $\tau \in \mathcal{T}$, and using the inequality

$$\begin{aligned} \frac{1}{2}\|(Y, Z \cdot M + N)\|_\rho^2 &\leq \|Y\|_\rho^2 \vee \|Z \cdot M + N\|_{BMO, \rho}^2 \\ &\leq \sup_{\tau \in \mathcal{T}} \left\| e^{\rho A_\tau} |Y_\tau|^2 + \mathbb{E}\left[\int_\tau^T e^{\rho A_s} \left(Z_s^\top d\langle M \rangle_s Z_s + d\langle N \rangle_s\right) \middle| \mathcal{F}_\tau\right] \right\|_\infty, \end{aligned}$$

we deduce by transferring $\frac{1}{4}\|Y\|_\rho^2$ to the left-hand side of (3.4) that

$$\begin{aligned} \|(Y, Z \cdot M + N)\|_\rho^2 &\leq 4\|\xi\|_{\infty, \rho}^2 + 32\|f(\cdot, 0, 0)\|_T^2 + \frac{1}{2}(\|y\|_\rho^2 + \|z \cdot M\|_{BMO, \rho}^2) + 32\tilde{g}^2 \|n\|_{BMO, \rho}^4 \\ &\leq 4\|\xi\|_{\infty, \rho}^2 + 32\|f(\cdot, 0, 0)\|_T^2 + \frac{1}{2}R^2 + 32\tilde{g}^2 R^4. \end{aligned}$$

Thanks to (3.2), $\tilde{g} = \frac{1}{8}$ and $R = \frac{1}{2}$, we verify from the above estimate that

$$\|(Y, Z, N)\|_\rho \leq R.$$

(ii). We prove $\mathbf{F} : (\mathbf{B}_R, \|\cdot\|_\rho) \rightarrow (\mathbf{B}_R, \|\cdot\|_\rho)$ is a contraction mapping. By (i), for $i = 1, 2$ and any $(y^i, z^i \cdot M + n^i) \in \mathbf{B}_R$, we have $(Y^i, Z^i \cdot M + N^i) := \mathbf{F}((y^i, z^i \cdot M + n^i)) \in \mathbf{B}_R$. For notational convenience we set $\delta y := y^1 - y^2$ and $\delta z, \delta n, \delta \langle n \rangle, \delta Y, \delta Z, \delta N, \delta \langle N \rangle$, etc. analogously. By the deductions in (i) with minor modifications, we obtain

$$\begin{aligned} \frac{1}{2}\|(\delta Y, \delta Z \cdot M + \delta N)\|_\rho^2 &\leq \frac{1}{8}(\|\delta y\|_\rho^2 + \|\delta z \cdot M\|_{BMO, \rho}^2) + \frac{1}{4}\|\delta Y\|_\rho^2 \\ &\quad + 4\tilde{g}^2 \sup_{\tau \in \mathcal{T}} \left\| \mathbb{E}\left[\int_\tau^T e^{\frac{\rho}{2} A_s} d|\delta \langle n \rangle_s| \middle| \mathcal{F}_\tau\right] \right\|_\infty^2. \end{aligned} \quad (3.5)$$

Kunita-Watanabe inequality and Cauchy-Schwartz inequality used to the last term gives

$$\begin{aligned} \mathbb{E}\left[\int_\tau^T e^{\frac{\rho}{2} A_s} d|\delta \langle n \rangle_s| \middle| \mathcal{F}_\tau\right]^2 &\leq \mathbb{E}\left[\int_\tau^T e^{\frac{\rho}{2} A_s} d\langle \delta n \rangle_s \middle| \mathcal{F}_\tau\right] \mathbb{E}\left[\int_\tau^T e^{\frac{\rho}{2} A_s} d\langle n^1 + n^2 \rangle_s \middle| \mathcal{F}_\tau\right] \\ &\leq \|\delta n\|_{BMO, \rho}^2 \cdot 2(\|n^1\|_{BMO, \rho}^2 + \|n^2\|_{BMO, \rho}^2) \\ &\leq \|\delta n\|_{BMO, \rho}^2 \cdot 4R^2, \end{aligned}$$

where the last inequality is due to $\|(y^i, z^i \cdot M + n^i)\|_\rho \leq R$, $i = 1, 2$. Hence (3.5) gives

$$\begin{aligned} \|(\delta Y, \delta Z \cdot M + \delta N)\|_\rho^2 &\leq \frac{1}{2}(\|\delta y\|_\rho^2 + \|\delta z \cdot M\|_{BMO, \rho}^2) + 64\tilde{g}^2 R^2 \|\delta n\|_{BMO, \rho}^2 \\ &\leq \left(\frac{1}{2} + 64\tilde{g}^2 R^2\right) \|(\delta y, \delta z \cdot M + \delta n)\|_\rho^2 \\ &\leq \frac{3}{4} \|(\delta y, \delta z \cdot M + \delta n)\|_\rho^2, \end{aligned}$$

i.e., $\mathbf{F} : (\mathbf{B}_R, \|\cdot\|_\rho) \rightarrow (\mathbf{B}_R, \|\cdot\|_\rho)$ is a contraction mapping. The existence of a solution in \mathbf{B}_R thus follows immediately from Banach fixed point theorem. Finally, since $\|\cdot\|$ is equivalent to $\|\cdot\|_\rho$ for \mathcal{B} , the solution also belongs to $(\mathcal{B}, \|\cdot\|)$.

□

From now on we denote $(\mathcal{B}, \|\cdot\|)$ by \mathcal{B} when there is no ambiguity. In the spirit of Tevzadze [31], we extend this existence result so as to allow any bounded data. To this end, for any \mathbb{Q} equivalent to \mathbb{P} we define $\mathcal{S}^\infty(\mathbb{Q})$ analogously to \mathcal{S}^∞ but under \mathbb{Q} . This notation also applies to other spaces.

Theorem 3.2 (Existence (ii)) *If (f, g, ξ) satisfies (A.1), then there exists a solution to (f, g, ξ) in \mathcal{B} .*

Proof. (i). We first show that it is equivalent to prove the existence result given $|g| \leq \frac{1}{8}$ \mathbb{P} -a.s. Suppose that g is bounded by a positive constant \tilde{g} , that is, $|g| \leq \tilde{g}$ \mathbb{P} -a.s. Observe that, for any $\theta > 0$, (Y, Z, N) is a solution to (f, g, ξ) if and only if $(\theta Y, \theta Z, \theta N)$ is a solution to $(f^\theta, g/\theta, \theta\xi)$, where $f^\theta(t, y, z) := \theta f(t, \frac{y}{\theta}, \frac{z}{\theta})$. Obviously f^θ verifies (A.1) with the same Lipschitz coefficients as f . If we set $\theta := 8\tilde{g}$, then $|g/\theta| \leq \frac{1}{8}$ \mathbb{P} -a.s. and hence satisfies the parametrization in Theorem 3.1 (existence (i)). Therefore, we can and do assume $|g| \leq \frac{1}{8}$ \mathbb{P} -a.s. without loss of generality.

(ii). Since $\|\xi\|_\infty + \| |f(\cdot, 0, 0)| |_T \|_\infty < +\infty$, we can find $n \in \mathbb{N}^+$ such that

$$\xi = \sum_{i=1}^n \xi^i, \quad f(t, 0, 0) = \sum_{i=1}^n f^i(t, 0, 0),$$

where, for each $i \leq n$, ξ^i is a \mathcal{F}_T -measurable random variable, $f^i : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ is $\text{Prog} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable and

$$\|\xi^i\|_\infty^2 + 8 \| |f^i(\cdot, 0, 0)| |_T \|_\infty^2 \leq \frac{1}{64} \exp \left(-\|A\| (8\beta^2 \|A\| + 8\gamma^2) \right).$$

Set $f'(t, y, z) := f(t, y, z) - f(t, 0, 0)$ and $(Y^0, Z^0 \cdot M + N^0) \in \mathcal{B}$ such that $\|(Y^0, Z^0 \cdot M + N^0)\| = 0$. Now we use a recursion argument in the following way for $i = 1, \dots, n$.

By Theorem 3.1, there exists a solution $(Y^i, Z^i \cdot M + \tilde{N}^i) \in \mathcal{B}(\mathbb{Q}^i)$ to the BSDE

$$\begin{aligned} Y_t^i &= \xi^i + \int_t^T \left(f^i(s, 0, 0) + f'(s, \sum_{j=0}^i Y_s^j, \sum_{j=0}^i Z_s^j) - f'(s, \sum_{j=0}^{i-1} Y_s^j, \sum_{j=0}^{i-1} Z_s^j) \right) dA_s \\ &\quad + \int_t^T g_s d\langle \tilde{N}^i \rangle_s - \int_t^T (Z_s^i dM_s + d\tilde{N}_s^i), \end{aligned}$$

where

$$\frac{d\mathbb{Q}^i}{d\mathbb{P}} := \mathcal{E} \left(2g \cdot \sum_{j=0}^{i-1} N^j \right)_T.$$

Note that the equivalent change of measure holds due to the fact that $N^j \in \mathcal{M}^{BMO}$ for $j \leq i-1$ and Theorem 2.3, Kazamaki [21]. By Girsanov transformation and Theorem 3.6, Kazamaki [21], $N^i := \tilde{N}^i + 2g \cdot \langle \tilde{N}^i, \sum_{j=0}^{i-1} N^j \rangle$ and $Z^i \cdot M$ belong to \mathcal{M}^{BMO} . This further implies $\langle N^i \rangle = \langle \tilde{N}^i \rangle$ and $N^i = \tilde{N}^i + 2g \cdot \langle N^i, \sum_{j=0}^{i-1} N^j \rangle$. Hence $(Y^i, Z^i \cdot M + N^i) \in \mathcal{B}$ solves

$$\begin{aligned} Y_t^i &= \xi^i + \int_t^T \left(f^i(s, 0, 0) + f'(s, \sum_{j=0}^i Y_s^j, \sum_{j=0}^i Z_s^j) - f'(s, \sum_{j=0}^{i-1} Y_s^j, \sum_{j=0}^{i-1} Z_s^j) \right) dA_s \\ &\quad + \int_t^T g_s d \left(\langle N^i \rangle_s + 2 \langle N^i, \sum_{j=0}^{i-1} N^j \rangle_s \right) - \int_t^T (Z_s^i dM_s + dN_s^i). \end{aligned}$$

Hence a recursion argument gives (Y^i, Z^i, N^i) for $i = 1, \dots, n$.

Define $Y := \sum_{i=0}^n Y^i$, $Z := \sum_{i=0}^n Z^i$ and $N := \sum_{i=0}^n N^i$. Clearly $(Y, Z \cdot M + N) \in \mathcal{B}$. We show (Y, Z, N) solves (f, g, ξ) . In view of the definition of f' , we sum up the above BSDEs to obtain

$$Y_t = \xi + \int_t^T \left((f(s, 0, 0) + f'(s, Y_s, Z_s)) dA_s + g_s d\langle N \rangle_s \right) - \int_t^T (\delta Z_s dM_s + d\delta N_s).$$

To conclude the proof we use $f'(s, Y_s, Z_s) := f(s, Y_s, Z_s) - f(s, 0, 0)$.

□

We continue to show that comparison theorem and hence uniqueness also hold given Lipschitz-continuity. Similar results in different settings can be found, e.g., in [24], [18], [26], [31].

Theorem 3.3 (Comparison) *Let $(Y, Z \cdot M + N), (Y', Z' \cdot M + N') \in \mathcal{S}^\infty \times \mathcal{M}^{BMO}$ be solutions to $(f, g, \xi), (f', g', \xi')$, respectively. If \mathbb{P} -a.s. for any $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$, $f(t, y, z) \leq f'(t, y, z)$, $g_t \leq g'_t$, $\xi \leq \xi'$ and (f, g, ξ) verifies (A.1), then \mathbb{P} -a.s. $Y \leq Y'$.*

Proof. Set $\delta Y := Y - Y'$ and $\delta Z, \delta N, \delta \langle N \rangle, \delta \xi$, etc. analogously. For any $\tau \in \mathcal{T}$, \mathbb{P} -a.s. $f \leq f'$ and $g \leq g'$ imply by Itô's formula that

$$\begin{aligned}
\delta Y_{t \wedge \tau} &= \delta Y_\tau + \int_{t \wedge \tau}^\tau (f(s, Y_s, Z_s) - f'(s, Y'_s, Z'_s)) dA_s + \int_{t \wedge \tau}^\tau g_s d\langle N \rangle_s - \int_{t \wedge \tau}^\tau g'_s d\langle N' \rangle_s \\
&\quad - \int_{t \wedge \tau}^\tau (\delta Z_s dM_s + d\delta N_s) \\
&\leq \delta Y_\tau + \int_{t \wedge \tau}^\tau (f(s, Y_s, Z_s) - f(s, Y'_s, Z'_s)) dA_s + \int_{t \wedge \tau}^\tau g'_s d\delta \langle N \rangle_s - \int_{t \wedge \tau}^\tau (\delta Z_s dM_s + d\delta N_s) \\
&= \delta Y_\tau + \int_{t \wedge \tau}^\tau (\beta_s \delta Y_s + (\lambda_s \gamma_s)^\top (\lambda_s \delta Z_s)) dA_s + \int_{t \wedge \tau}^\tau g'_s d\delta \langle N \rangle_s - \int_{t \wedge \tau}^\tau (\delta Z_s dM_s + d\delta N_s),
\end{aligned} \tag{3.6}$$

where β (\mathbb{R} -valued) and γ (\mathbb{R}^d -valued) are defined by

$$\begin{aligned}
\beta_s &:= \mathbb{I}_{\{\delta Y_s \neq 0\}} \frac{f(s, Y_s, Z_s) - f(s, Y'_s, Z_s)}{\delta Y_s}, \\
\gamma_s &:= \mathbb{I}_{\{\lambda_s \delta Z_s \neq 0\}} \frac{(f(s, Y'_s, Z_s) - f(s, Y'_s, Z'_s)) \delta Z_s}{|\lambda_s \delta Z_s|^2},
\end{aligned}$$

and $\mathbf{0} := (0, \dots, 0)^\top$. Note that γ can be seen as defined in terms of discrete gradient. By (A.1), β , and $\int_0^\tau \gamma_s^\top d\langle M \rangle_s \gamma_s$ are bounded processes, hence $\gamma \cdot M \in \mathcal{M}^{BMO}$. Given these facts we use a change of measure to attain the comparison result. To this end, we define a BMO martingale

$$\Lambda := \gamma \cdot M + g' \cdot (N + N').$$

In view of Theorem 2.3 and Theorem 3.6, Kamazaki [21], we define

$$\frac{d\mathbb{Q}}{d\mathbb{P}} := \mathcal{E}(\Lambda)_T.$$

Hence $\delta N - g' \cdot \delta \langle N \rangle$ and $\delta Z \cdot M - (\gamma^\top \lambda^\top \lambda \delta Z) \cdot A$ belong to $\mathcal{M}^{BMO}(\mathbb{Q})$. Therefore, (3.6) and \mathbb{P} -a.s. $\delta \xi \leq 0$ give

$$\begin{aligned}
\delta Y_t &\leq \mathbb{E}^\mathbb{Q}[\delta \xi | \mathcal{F}_t] + \mathbb{E}^\mathbb{Q} \left[\int_t^T \beta_s \delta Y_s dA_s \middle| \mathcal{F}_t \right] \\
&\leq \mathbb{E}^\mathbb{Q} \left[\int_t^T \beta_s \delta Y_s dA_s \middle| \mathcal{F}_t \right].
\end{aligned}$$

Hence we obtain by Gronwall's lemma that \mathbb{P} -a.s. $\delta Y_t \leq 0$. Finally by the continuity of Y and Y' , we conclude that \mathbb{P} -a.s. $Y \leq Y'$.

□

As a byproduct, we obtain the following existence and uniqueness result.

Corollary 3.4 (Uniqueness) *If (f, g, ξ) satisfies (A.1), then there exists a unique solution in \mathcal{B} .*

Proof. This is immediate from Theorem 3.2 (existence (ii)) and Theorem 3.3 (comparison theorem). □

3.3 Bounded Solutions to Quadratic BSDEs

In this section, we prove a general monotone stability result for quadratic BSDEs. Let us recall that Morlais [26] uses a stability-type argument for the existence result after performing an exponential transform which eliminates $g \cdot \langle N \rangle$. But a direct stability result is not studied. Our work fills this gap.

Secondly, as a byproduct, we construct a bounded solution via regularization through Lipschitz-quadratic BSDEs studied in Section 3.3. This procedure is also called *Lipschitz-quadratic regularization* in the following context. To this end we give the assumptions for the whole section.

Assumption (A.2) There exist $\beta \geq 0$, $\gamma > 0$, an \mathbb{R}^+ -valued Prog-measurable process α and a continuous nondecreasing function $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $\varphi(0) = 0$ such that $\|\xi\|_\infty + \|\alpha|_T\|_\infty < +\infty$ and \mathbb{P} -a.s.

- (i) for any $t \in [0, T]$, $(y, z) \mapsto f(t, y, z)$ is continuous;
- (ii) f is monotonic at $y = 0$, i.e., for any $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$,

$$\operatorname{sgn}(y)f(t, y, z) \leq \alpha_t + \alpha_t \beta |y| + \frac{\gamma}{2} |\lambda_t z|^2;$$

- (iii) for any $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$,

$$|f(t, y, z)| \leq \alpha_t + \alpha_t \varphi(|y|) + \frac{\gamma}{2} |\lambda_t z|^2.$$

We continue as before to call $(\xi, |\alpha|_T)$ the *data*. (A.2)(ii) allows one to get rid of the linear growth in y which is required by Kobylanski [22] and Morlais [26]. Assumption of this type for quadratic framework is motivated by Briand and Hu [9]. Secondly, our results don't rely on the specific choice of φ . Hence the growth condition in y can be arbitrary as long as (A.2)(i)(ii) hold.

Given (A.2), we first prove an a priori estimate. In order to treat $\langle Z \cdot M \rangle$ and $g \cdot \langle N \rangle$ more easily, we assume \mathbb{P} -a.s. $|g| \leq \frac{\gamma}{2}$ for the rest of this chapter.

Lemma 3.5 (A Priori Estimate) *If (f, g, ξ) satisfies (A.2) and $(Y, Z \cdot M + N) \in \mathcal{S}^\infty \times \mathcal{M}$ is a solution to (f, g, ξ) , then*

$$\|Y\| \leq \|e^{\beta|\alpha|_T}(|\xi| + |\alpha|_T)\|_\infty$$

and

$$\|Z \cdot M + N\|_{BMO} \leq c_b,$$

where c_b is a constant only depending on $\beta, \gamma, \|\xi\|_\infty, \|\alpha\|_T$.

Proof. Set $u(x) := \frac{\exp(\gamma x) - 1 - \gamma x}{\gamma^2}$. The following auxiliary results will be useful: $u(x) \geq 0, u'(x) \geq 0$ and $u''(x) \geq 1$ for $x \geq 0$; $u(|\cdot|) \in \mathcal{C}^2(\mathbb{R})$ and $u''(x) = \gamma u'(x) + 1$. For any $\tau, \sigma \in \mathcal{T}$, Itô's formula yields

$$u(|Y_{\tau \wedge \sigma}|) = u(|Y_\sigma|) + \int_{\tau \wedge \sigma}^\sigma u'(|Y_s|) \operatorname{sgn}(Y_s) dY_s - \frac{1}{2} \int_{\tau \wedge \sigma}^\sigma u''(|Y_s|) (Z_s^\top d\langle M \rangle_s Z_s + d\langle N \rangle_s).$$

By (A.2)(ii),

$$\operatorname{sgn}(Y_s) f(s, Y_s, Z_s) \leq \alpha_s + \alpha_s \beta |Y_s| + \frac{\gamma}{2} |\lambda_s Z_s|^2.$$

Note that $\frac{\gamma}{2} u'(|Y_s|) - \frac{1}{2} u''(|Y_s|) = -\frac{1}{2}$, $g_s u'(|Y_s|) - \frac{1}{2} u''(|Y_s|) \leq -\frac{1}{2}$. and $u'(|Y_s|) \leq \frac{e^{\gamma \|Y\|}}{\gamma}$. Hence, using these facts to the above equality yields

$$\begin{aligned} \frac{1}{2} \int_{\tau \wedge \sigma}^\sigma (Z_s^\top d\langle M \rangle_s Z_s + d\langle N \rangle_s) &\leq \frac{e^{\gamma \|Y\|}}{\gamma^2} + \int_{\tau \wedge \sigma}^\sigma u'(|Y_s|) (\alpha_s + \alpha_s \beta |Y_s|) dA_s \\ &\quad - \int_{\tau \wedge \sigma}^\sigma u'(|Y_s|) \operatorname{sgn}(Y_s) (Z_s dM_s + dN_s). \end{aligned}$$

To eliminate the local martingale, we replace σ by its localizing sequence and use Fatou's lemma to the left-hand side. Since Y^* and $|\alpha|_T$ are bounded random variables, the right-hand side has a uniform constant upper bound. Hence, we have

$$\frac{1}{2} \mathbb{E}[\langle Z \cdot M + N \rangle_{\tau, T} | \mathcal{F}_\tau] \leq \frac{e^{\gamma \|Y\|}}{\gamma^2} + \frac{e^{\gamma \|Y\|}}{\gamma} (1 + \beta \|Y\|) \|\alpha\|_T. \quad (3.7)$$

Now we turn to the estimate for Y . We fix $s \in [0, T]$ and for $t \in [s, T]$, set

$$H_t := \exp \left(\gamma e^{\beta |\alpha|_{s, t}} |Y_t| + \gamma \int_s^t e^{\beta |\alpha|_{s, u}} \alpha_u dA_u \right).$$

We claim that H is a submartingale. By Tanaka's formula,

$$d|Y_t| = \operatorname{sgn}(Y_t) (Z_t dM_t + dN_t) - \operatorname{sgn}(Y_t) (f(t, Y_t, Z_t) dA_t + g_t d\langle N \rangle_t) + dL_t^0(Y),$$

where $L^0(Y)$ is the local time of Y at 0. Hence, Itô's formula yields

$$\begin{aligned} dH_t = & \gamma H_t e^{\beta|\alpha|_{s,t}} \left[\operatorname{sgn}(Y_t) (Z_t dM_t + dN_t) \right. \\ & + \left(-\operatorname{sgn}(Y_t) f(t, Y_t, Z_t) + \alpha_t + \alpha_t \beta |Y_t| + \frac{\gamma}{2} e^{\beta|\alpha|_{s,t}} |\lambda_t Z_t|^2 \right) dA_t \\ & \left. + \left(-\operatorname{sgn}(Y_t) g_t + \frac{\gamma}{2} e^{\beta|\alpha|_{s,t}} \right) d\langle N \rangle_t + dL_t^0(Y) \right]. \end{aligned}$$

By (A.2)(ii) and $|g_t| \leq \frac{\gamma}{2}$ again, $(H_t)_{t \in [s, T]}$ is a bounded submartingale. Hence,

$$|Y_s| \leq \frac{1}{\gamma} \ln \mathbb{E}[H_T | \mathcal{F}_s].$$

Thanks to the boundedness, we have

$$\|Y\| \leq \|e^{\beta|\alpha|_T} (|\xi| + |\alpha|_T)\|_\infty.$$

Finally we come back to (3.7) and obtain the estimate for $Z \cdot M + N$.

□

Given the norm bound in Lemma 3.5, we turn to the main result of this section: monotone stability result. Later, as an immediate application, we prove an existence result for quadratic BSDEs by Lipschitz-quadratic regularization. To start we recall that \mathcal{M}^2 equipped with the norm $\|\widetilde{M}\|_{\mathcal{M}^2} := \mathbb{E}[\langle \widetilde{M} \rangle_T]^{\frac{1}{2}}$ for $\widetilde{M} \in \mathcal{M}^2$ is a Hilbert space.

Theorem 3.6 (Monotone Stability) *Let $(f^n, g^n, \xi^n)_{n \in \mathbb{N}^+}$ satisfy (A.2) associated with $(\alpha, \beta, \gamma, \varphi)$, and $(Y^n, Z^n \cdot M + N^n)$ be their solutions in \mathcal{B} , respectively. Assume*

- (i) Y^n is monotonic in n and $\xi^n - \xi \rightarrow 0$ \mathbb{P} -a.s. with $\sup_n \|\xi^n\|_\infty < +\infty$;
- (ii) \mathbb{P} -a.s. for any $t \in [0, T]$, $g_t^n - g_t \rightarrow 0$;
- (iii) \mathbb{P} -a.s. for any $t \in [0, T]$ and $y^n \rightarrow y, z^n \rightarrow z$, $f^n(t, y^n, z^n) \rightarrow f(t, y, z)$.

Then there exists $(Y, Z \cdot M + N) \in \mathcal{B}$ such that Y^n converges to Y \mathbb{P} -a.s. uniformly on $[0, T]$ and $(Z^n \cdot M + N^n)$ converges to $(Z \cdot M + N)$ in \mathcal{M}^2 as n goes to $+\infty$. Moreover, (Y, Z, N) solves (f, g, ξ) .

Proof. Without loss of generality we only consider Y^n to be increasing in n . By Lemma 3.5 (a priori estimate),

$$\sup_n \|Y^n\| + \sup_n \|Z^n \cdot M + N^n\|_{BMO} \leq c_b, \quad (3.8)$$

where c_b is a constant only depending on $\beta, \gamma, \sup_n \|\xi^n\|_\infty, \|\alpha\|_T$. We rely intensively on the boundedness result in (3.8) to derive the limit.

(i). We prove the convergence of the solutions. Due to (3.8), there exists a bounded monotone limit $Y_t := \lim_n Y_t^n$, a subsequence indexed by $\{n_k\}_{k \in \mathbb{N}^+} \subseteq \mathbb{N}^+$ and $Z \cdot M + N \in \mathcal{M}^2$ such that $Z^{n_k} \cdot M + N^{n_k}$ converges weakly in \mathcal{M}^2 to $Z \cdot M + N$ as k goes to $+\infty$. The task is to show $Z \cdot M + N$ is the \mathcal{M}^2 -limit of the whole sequence. To this end we define $u(x) := \frac{\exp(8\gamma x) - 8\gamma x - 1}{64\gamma^2}$. Recall that $u(x) \geq 0, u'(x) \geq 0$ and $u''(x) \geq 0$ for $x \geq 0$; $u \in \mathcal{C}^2(\mathbb{R})$ and $u''(x) = 8\gamma u'(x) + 1$. For any $m \in \{n_k\}_{k \in \mathbb{N}^+}, n \in \mathbb{N}^+$ with $m \geq n$, define $\delta Y^{m,n} := Y^m - Y^n, \delta Y^n := Y - Y^n$ and $\delta Z^{m,n}, \delta Z^n, \delta N^{m,n}, \delta N^n$, etc. analogously. By Itô's formula,

$$\begin{aligned} \mathbb{E}[u(\delta Y_0^{m,n})] - \mathbb{E}[u(\delta \xi^{m,n})] &= \mathbb{E}\left[\int_0^T u'(\delta Y_s^{m,n})(f^m(s, Y_s^m, Z_s^m) - f^n(s, Y_s^n, Z_s^n))dA_s\right] \\ &\quad + \mathbb{E}\left[\int_0^T u'(\delta Y_s^{m,n})(g_s^m d\langle N^m \rangle_s - g_s^n d\langle N^n \rangle_s)\right] \\ &\quad - \frac{1}{2}\mathbb{E}\left[\int_0^T u''(\delta Y_s^{m,n})\left((\delta Z_s^{m,n})^\top d\langle M \rangle_s(\delta Z_s^{m,n}) + d\langle \delta N^{m,n} \rangle_s\right)\right]. \end{aligned} \tag{3.9}$$

Since f^m and f^n verify (A.2) associated with $(\alpha, \beta, \gamma, \varphi)$, we have

$$\begin{aligned} |f^m(s, Y_s^m, Z_s^m) - f^n(s, Y_s^n, Z_s^n)| &\leq \alpha'_s + \frac{\gamma}{2}|\lambda_s Z_s^m|^2 + \frac{\gamma}{2}|\lambda_s Z_s^n|^2 \\ &\leq \alpha'_s + \frac{3\gamma}{2}(|\lambda_s \delta Z_s^{m,n}|^2 + |\lambda_s \delta Z_s^n|^2 + |\lambda_s Z_s|^2) + \gamma(|\lambda_s \delta Z_s^n|^2 + |\lambda_s Z_s|^2) \\ &\leq \alpha'_s + \frac{3\gamma}{2}|\lambda_s \delta Z_s^{m,n}|^2 + \frac{5\gamma}{2}(|\lambda_s \delta Z_s^n|^2 + |\lambda_s Z_s|^2), \end{aligned}$$

where

$$\alpha'_s := 2\alpha_s(1 + \varphi(c_b)) \geq 2\alpha_s + \alpha_s\varphi(|Y_s^n|) + \alpha_s\varphi(|Y_s^m|).$$

Moreover,

$$\begin{aligned} g^m d\langle N^m \rangle - g^n d\langle N^n \rangle &\ll \frac{\gamma}{2}d\langle N^m \rangle + \frac{\gamma}{2}d\langle N^n \rangle \\ &\ll \frac{3\gamma}{2}d\langle \delta N^{m,n} \rangle + \frac{5\gamma}{2}(d\langle \delta N^n \rangle + d\langle N \rangle). \end{aligned}$$

Plugging the above inequalities into (3.9), we deduce that

$$\begin{aligned}
& \mathbb{E} \left[\int_0^T \left(\frac{1}{2} u'' - \frac{3\gamma}{2} u' \right) (\delta Y_s^{m,n}) |\lambda_s \delta Z_s^{m,n}|^2 dA_s \right] + \mathbb{E} \left[\int_0^T \left(\frac{1}{2} u'' - \frac{3\gamma}{2} u' \right) (\delta Y_s^{m,n}) d\langle \delta N^{m,n} \rangle_s \right] \\
& \leq \mathbb{E} [u(\delta \xi^{m,n})] + \mathbb{E} \left[\int_0^T u'(\delta Y_s^{m,n}) \left(\alpha'_s + \frac{5\gamma}{2} (|\lambda_s \delta Z_s^n|^2 + |\lambda_s Z_s|^2) \right) dA_s \right] \\
& + \mathbb{E} \left[\int_0^T u'(\delta Y_s^{m,n}) \frac{5\gamma}{2} (d\langle \delta N^n \rangle_s + d\langle N \rangle_s) \right] \tag{3.10}
\end{aligned}$$

Due to the weak convergence result and convexity of $z \mapsto |z|^2$, $N \mapsto \langle N \rangle$, we obtain

$$\begin{aligned}
& \mathbb{E} \left[\int_0^T \left(\frac{1}{2} u'' - \frac{3\gamma}{2} u' \right) (\delta Y_s^n) |\lambda_t Z_s^n|^2 dA_s \right] \leq \liminf_m \mathbb{E} \left[\int_0^T \left(\frac{1}{2} u'' - \frac{3\gamma}{2} u' \right) (\delta Y_s^{m,n}) |\lambda_t Z_s^{m,n}|^2 dA_s \right], \\
& \mathbb{E} \left[\int_0^T \left(\frac{1}{2} u'' - \frac{3\gamma}{2} u' \right) (\delta Y_s^n) d\langle \delta N^n \rangle_s \right] \leq \liminf_m \mathbb{E} \left[\int_0^T \left(\frac{1}{2} u'' - \frac{3\gamma}{2} u' \right) (\delta Y_s^{m,n}) d\langle \delta N^{m,n} \rangle_s \right].
\end{aligned}$$

We then come back to (3.10) and send m to $+\infty$ along $\{n_k\}_{k \in \mathbb{N}^+}$. Taking the above inequalities into account and using $u'(\delta Y_s^{m,n}) \leq u'(\delta Y_s^n)$ to the right-hand side, (3.10) becomes

$$\begin{aligned}
& \mathbb{E} \left[\int_0^T \left(\frac{1}{2} u'' - \frac{3\gamma}{2} u' \right) (\delta Y_s^n) |\lambda_s Z_s^n|^2 dA_s \right] \\
& + \mathbb{E} \left[\int_0^T \left(\frac{1}{2} u'' - \frac{3\gamma}{2} u' \right) (\delta Y_s^n) d\langle \delta N^n \rangle_s \right] \\
& \leq \mathbb{E} [u(\delta \xi^n)] + \mathbb{E} \left[\int_0^T u'(\delta Y_s^n) \left(\alpha'_s + \frac{5\gamma}{2} (|\lambda_s \delta Z_s^n|^2 + |\lambda_s Z_s|^2) \right) dA_s \right] \\
& + \frac{5\gamma}{2} \mathbb{E} \left[\int_0^T u'(\delta Y_s^n) (d\langle \delta N^n \rangle_s + d\langle N \rangle_s) \right]. \tag{3.11}
\end{aligned}$$

Since $u''(x) - 8\gamma u'(x) = 1$, rearranging terms give

$$\begin{aligned}
& \frac{1}{2} E[(\delta N_T^n)^2] + \frac{1}{2} \mathbb{E} \left[\int_0^T |\lambda_s \delta Z_s^n|^2 dA_s \right] \\
& \leq \mathbb{E} [u(\delta \xi^n)] + \mathbb{E} \left[\int_0^T u'(\delta Y_s^n) \left(\alpha'_s + \frac{5\gamma}{2} |\lambda_s Z_s|^2 \right) dA_s \right] + \frac{5\gamma}{2} \mathbb{E} \left[\int_0^T u'(\delta Y_s^n) d\langle N \rangle_s \right]. \tag{3.12}
\end{aligned}$$

Finally by sending n to $+\infty$ and dominated convergence we deduce the convergence.

(ii). We prove $(Y, Z \cdot M + N) \in \mathcal{B}$ and solves (f, g, ξ) . Here we rely on the same arguments as in Kobylanski [22] or Morlais [26] and omit the details here. In addition, we need to prove the *u.c.p* convergence of $g^n \cdot \langle N^n \rangle$, which holds if

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\left| \int_0^\cdot (g_s^n d\langle N^n \rangle_s - g_s d\langle N \rangle_s) \right|^* \right] = 0.$$

Indeed, by Kunita-Watanabe inequality and Cauchy-Schwartz inequality,

$$\begin{aligned}
\mathbb{E} \left[\left| \int_0^\cdot (g_s^n d\langle N^n \rangle_s - g_s d\langle N \rangle_s) \right|^* \right] &= \mathbb{E} \left[\left| \int_0^\cdot \left(g_s^n d(\langle N^n \rangle_s - \langle N \rangle_s) + (g_s^n - g_s) d\langle N \rangle_s \right) \right|^* \right] \\
&\leq \frac{\gamma}{2} \mathbb{E} [\langle N^n - N \rangle_T]^{\frac{1}{2}} \mathbb{E} [\langle N^n + N \rangle_T]^{\frac{1}{2}} + \mathbb{E} \left[\left| \int_0^\cdot (g_s^n - g_s) d\langle N \rangle_s \right|^* \right] \\
&\leq \gamma c_b \mathbb{E} [\langle N^n - N \rangle_T]^{\frac{1}{2}} + \mathbb{E} \left[\int_0^T |g_s^n - g_s| d\langle N \rangle_s \right].
\end{aligned}$$

We then conclude by \mathcal{M}^2 -convergence of N^n and dominated convergence used to the second term. Finally $Z \cdot M + N \in \mathcal{M}^{BMO}$ by Lemma 3.5 (a priori estimate).

For decreasing Y^n , we take $m \in \mathbb{N}^+, n \in \{n_k\}_{k \in \mathbb{N}^+}$ with $n \geq m$ and conclude with exactly the same arguments.

□

There are several major improvements compared to existing monotone stability results. First of all, in contrast to Kobylanski [22] and Morlais [26], we get rid of linear growth in y by merely assuming (A.2), and allow g to be any bounded process. Secondly, we treat the convergence in a more direct and general way than Morlais [26].

Another advantage concerns the existence result. Thanks to Section 3.2 and Theorem 3.6, we are able to perform a Lipschitz-quadratic regularization where exponential transform to eliminate $g \cdot \langle N \rangle$ is no longer needed; this is in contrast to Morlais [26]. This also helps to prove the existence of unbounded solutions with fewer assumptions; see Section 3.4.

Proposition 3.7 (Existence) *If (f, g, ξ) satisfy (A.2), then there exists a solution in \mathcal{B} .*

Proof. We use a double approximation procedure and use Theorem 3.6 (monotone stability) to take the limit. Define

$$\begin{aligned}
f^{n,k}(t, y, z) &:= \inf_{y', z'} \left\{ f^+(t, y', z') + n|y - y'| + n|\lambda_t(z - z')| \right\} \\
&\quad - \inf_{y', z'} \left\{ f^-(t, y', z') + k|y - y'| + k|\lambda_t(z - z')| \right\}.
\end{aligned}$$

By Lepeltier and San Martin [23], $f^{n,k}$ is Lipschitz-continuous in (y, z) ; as k goes to $+\infty$, $f^{n,k}$ converges increasingly uniformly on compact sets to a limit denoted by $f^{n,\infty}$; as n goes to $+\infty$, $f^{n,\infty}$ converges increasingly uniformly on compact sets to f .

By Corollary 3.4, there exists a unique solution $(Y^{n,k}, Z^{n,k} \cdot M + N^{n,k}) \in \mathcal{B}$ to $(f^{n,k}, g, \xi)$; by Theorem 3.3 (comparison theorem), $Y^{n,k}$ is increasing in n and decreasing in k , and is uniformly bounded due to Lemma 3.5 (a priori estimate). We then fix n and

use Theorem 3.6 to the sequence indexed by k to obtain a solution $(Y^n, Z^n \cdot M + N^n) \in \mathcal{B}$ to $(f^{n,\infty}, g, \xi)$. Due to the \mathbb{P} -a.s. uniform convergence of $Y^{n,k}$ we can pass the comparison property to Y^n . We use Theorem 3.6 again to conclude.

□

Remark. In contrast to Kobylanski [22], the existence of a maximal or minimal solution is not available (yet) given (A.1) as the double approximation procedure makes the comparison between solutions impossible.

There is also a rich literature on the uniqueness of a bounded solution to quadratic BSDEs; see, e.g., [22], [24], [18], [26]. Roughly speaking, they essentially rely a type of locally Lipschitz-continuity and use a change of measure analogously to Section 3.2. The proof in our setting is exactly the same and hence omitted to save pages.

To end this section we briefly present various structure conditions used in different situations.

Assumption (A.2') There exist $\beta \geq 0, \gamma > 0$, an \mathbb{R}^+ -valued Prog-measurable process α , and a continuous nondecreasing function $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $\varphi(0) = 0$ such that \mathbb{P} -a.s.

- (i) for any $t \in [0, T]$, $(y, z) \mapsto f(t, y, z)$ is continuous;
- (ii) f is monotonic at $y = 0$, i.e., for any $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$,

$$\operatorname{sgn}(y)f(t, y, z) \leq \alpha_t + \beta|y| + \frac{\gamma}{2}|\lambda_t z|^2;$$

- (iii) for any $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$,

$$|f(t, y, z)| \leq \alpha_t + \varphi(|y|) + \frac{\gamma}{2}|\lambda_t z|^2.$$

Given bounded data, (A.2') implies (A.2). Indeed,

$$\begin{aligned} \operatorname{sgn}(y)f(t, y, z) &\leq \alpha_t \vee 1 + (\alpha_t \vee 1)\beta|y| + \frac{\gamma}{2}|\lambda_t z|^2, \\ |f(t, y, z)| &\leq \alpha_t \vee 1 + (\alpha_t \vee 1)\varphi(|y|) + \frac{\gamma}{2}|\lambda_t z|^2. \end{aligned}$$

Hence (A.2') verifies (A.2) associated with $(\alpha \vee 1, \beta, \gamma, \varphi)$. However, given unbounded data, (A.2') appears to be more natural and convenient. This will be discussed in detail in Section 3.4.

In particular situations where the estimate for $\int_0^T |f(s, Y_s, Z_s)| dA_s$ is needed, e.g., in analysis of measure change (see Section 3.5) or the monotone stability of quadratic semimartingales (see Chapter 4), there has to be a linear growth in y , i.e.,

Assumption (A.2'') There exist $\beta \geq 0$, $\gamma > 0$, an \mathbb{R}^+ -valued Prog-measurable process α such that \mathbb{P} -a.s.

- (i) for any $t \in [0, T]$, $(y, z) \mapsto f(t, y, z)$ is continuous;
- (ii) for any $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$,

$$|f(t, y, z)| \leq \alpha_t + \beta|y| + \frac{1}{2}|\lambda_t z|^2.$$

Indeed, (A.2'') enables one to obtain the estimate for $\int_0^T |f(s, Y_s, Z_s)| dA_s$ via

$$\int_0^T |f(s, Y_s, Z_s)| dA_s \leq |\alpha|_T + \beta \|A\| Y^* + \frac{\gamma}{2} \langle Z \cdot M \rangle_T.$$

3.4 Unbounded Solutions to Quadratic BSDEs

This section extends Section 3.2, 3.3 to unbounded solutions. We prove an existence result and later show that the uniqueness holds given convexity assumption as an additional requirement. We point out that similar results have been obtained by Mocha and Westray [25], but our results rely on much fewer assumptions and are more natural. Analogously to section 3.3, we give an a priori estimate in the first step. We keep in mind that \mathbb{P} -a.s. $|g| \leq \frac{\gamma}{2}$ throughout our study.

Lemma 3.8 (A priori estimate) *If (f, g, ξ) satisfies (A.2') and $(Y, Z \cdot M + N) \in \mathcal{S} \times \mathcal{M}$ is a solution to (f, g, ξ) such that the process*

$$\exp \left(\gamma e^{\beta A_T} |Y| + \gamma \int_0^T e^{\beta A_s} \alpha_s dA_s \right)$$

is of class \mathcal{D} , then

$$|Y_s| \leq \frac{1}{\gamma} \ln \mathbb{E} \left[\exp \left(\gamma e^{\beta A_{s,T}} |\xi| + \gamma \int_s^T e^{\beta A_{s,u}} \alpha_u dA_u \right) \middle| \mathcal{F}_s \right]. \quad (3.13)$$

Proof. We fix $s \in [0, T]$, and for $t \in [s, T]$, set

$$H_t := \exp \left(\gamma e^{\beta A_{s,t}} |Y_t| + \gamma \int_s^t e^{\beta A_{s,u}} \alpha_u dA_u \right). \quad (3.14)$$

We claim that H is a local submartingale. Indeed, by Tanaka's formula

$$d|Y_t| = \text{sgn}(Y_t) (Z_t dM_t + dN_t) - \text{sgn}(Y_t) (f(t, Y_t, Z_t) dA_t + g_t d\langle N \rangle_t) + dL_t^0(Y),$$

where $L^0(Y)$ is the local time of Y at 0. Hence, Itô's formula yields

$$\begin{aligned} dH_t &= \gamma H_t e^{\beta A_{s,t}} \left[\operatorname{sgn}(Y_t) (Z_t dM_t + dN_t) \right. \\ &\quad + \left(-\operatorname{sgn}(Y_t) f(t, Y_t, Z_t) + \alpha_t + \beta |Y_t| + \frac{\gamma}{2} e^{\beta A_{s,t}} |\lambda_t Z_t|^2 \right) dA_t \\ &\quad \left. + \left(-\operatorname{sgn}(Y_t) g_t + \frac{\gamma}{2} e^{\beta A_{s,t}} \right) d\langle N \rangle_t + dL_t^0(Y) \right]. \end{aligned}$$

By (A.2')(ii), H is a local submartingale. To eliminate the local martingale part, we replace τ by its localizing sequence on $[s, T]$, denoted by $\{\tau_n\}_{n \in \mathbb{N}^+}$. Therefore,

$$\begin{aligned} |Y_s| &\leq \frac{1}{\gamma} \ln \mathbb{E} [H_{T \wedge \tau_n} | \mathcal{F}_s] \\ &\leq \frac{1}{\gamma} \ln \mathbb{E} \left[\exp \left(\gamma e^{\beta A_{s, T \wedge \tau_n}} |Y_{T \wedge \tau_n}| + \gamma \int_s^{T \wedge \tau_n} e^{\beta A_{s,u}} \alpha_u dA_u \right) \middle| \mathcal{F}_s \right]. \end{aligned}$$

Finally by class \mathcal{D} property we conclude by sending n to $+\infty$. □

We then know from Lemma 3.8 that exponential moments integrability on $|\xi| + |\alpha|_T$ is a natural requirement for the existence result.

Remark. (A.2') addresses the issue of integrability better than (A.2). To show this, let us assume (A.2). We then deduce from Lemma 3.5 and corresponding class \mathcal{D} property that

$$|Y_s| \leq \frac{1}{\gamma} \ln \mathbb{E} \left[\exp \left(\gamma e^{\beta |\alpha|_{s,T}} |\xi| + \gamma \int_s^T e^{\beta |\alpha|_{s,u}} \alpha_u dA_u \right) \middle| \mathcal{F}_s \right]. \quad (3.15)$$

Obviously, in (3.15), even exponential moments integrability is not sufficient to ensure the well-posedness of the a priori estimate. For more discussions on the choice of structure conditions, the reader shall refer to Mocha and Westray [25].

Motivated by the above discussions, we prove an existence result given (A.2') and exponential moments integrability. Analogously to Theorem 3.7, we use a Lipschitz-quadratic regularization and take the limit by the monotone stability result in Section 3.3. The a priori bound for Y obtained in Lemma 3.8 is also crucial to the construction of an unbounded solution.

Theorem 3.9 (Existence) *If (f, g, ξ) satisfies (A.2') and $e^{\beta A_T} (|\xi| + |\alpha|_T)$ has exponential moment of order γ , i.e.,*

$$\mathbb{E} \left[\exp \left(\gamma e^{\beta A_T} (|\xi| + |\alpha|_T) \right) \right] < +\infty,$$

then there exists a solution verifying (3.13).

Proof. We introduce the notations used throughout the proof. Define the process

$$X_t := \frac{1}{\gamma} \ln \mathbb{E} \left[\exp \left(\gamma e^{\beta A_T} (|\xi| + |\alpha|_T) \right) \middle| \mathcal{F}_t \right].$$

Obviously X is continuous by the continuity of the filtration. For $m, n \in \mathbb{N}^+$, set

$$\begin{aligned} \tau_m &:= \inf \{t \geq 0 : |\alpha|_t + X_t \geq m\} \wedge T, \\ \sigma_n &:= \inf \{t \geq 0 : |\alpha|_t \geq n\} \wedge T. \end{aligned}$$

It then follows from the continuity of X and $|\alpha|$ that τ_m and σ_n increase stationarily to T as m, n goes to $+\infty$, respectively. To apply a double approximation procedure, we define

$$\begin{aligned} f^{n,k}(t, y, z) &:= \mathbb{I}_{\{t \leq \sigma_n\}} \inf_{y', z'} \{f^+(t, y', z') + n|y - y'| + n|\lambda_t(z - z')|\} \\ &\quad - \mathbb{I}_{\{t \leq \sigma_k\}} \inf_{y', z'} \{f^-(t, y', z') + k|y - y'| + k|\lambda_t(z - z')|\}, \end{aligned}$$

and $\xi^{n,k} := \xi^+ \wedge n - \xi^- \wedge k$.

Before proceeding to the proof we give some useful facts. By Lepeltier and San Martin [23], $f^{n,k}$ is Lipschitz-continuous in (y, z) ; as k goes to $+\infty$, $f^{n,k}$ converges decreasingly uniformly on compact sets to a limit denoted by $f^{n,\infty}$; as n goes to $+\infty$, $f^{n,\infty}$ converges increasingly uniformly on compact sets to F . Moreover, $\|f^{n,k}(\cdot, 0, 0)\|_T$ and $\xi^{n,k}$ are bounded.

Hence, by Corollary 3.4, there exists a unique solution $(Y^{n,k}, Z^{n,k} \cdot M + N^{n,k}) \in \mathcal{B}$ to $(f^{n,k}, g, \xi^{n,k})$; by Theorem 3.3 (comparison theorem), $Y^{n,k}$ is increasing in n and decreasing in k . Analogously to Proposition 3.7, we wish to take the limit by Theorem 3.6 (monotone stability).

However, $\|f^{n,k}(\cdot, 0, 0)\|_T$ and $\xi^{n,k}$ are not uniformly bounded in general. To overcome this difficulty, we use Lemma 3.8 (a priori estimate) and work on random interval where $Y^{n,k}$ and $\|f^{n,k}(\cdot, 0, 0)\|$ are uniformly bounded. This is the motivation to introduce X and τ_m . To be more precise, the localization procedure is as follows.

Note that $(f^{n,k}, g, \xi^{n,k})$ verifies (A.2') associated with $(\alpha, \beta, \gamma, \varphi)$. $Y^{n,k}$ being bounded implies that it is of class \mathcal{D} . Hence from Lemma 3.8 we have

$$\begin{aligned} |Y_t^{n,k}| &\leq \frac{1}{\gamma} \ln \mathbb{E} \left[\exp \left(\gamma e^{\beta A_{t,T}} |\xi^{n,k}| + \gamma \int_t^T e^{\beta A_{t,s}} \alpha_s \mathbb{I}_{\{s \leq \sigma_n \wedge \sigma_k\}} dA_s \right) \middle| \mathcal{F}_t \right] \\ &\leq \frac{1}{\gamma} \ln \mathbb{E} \left[\exp \left(\gamma e^{\beta A_{t,T}} |\xi| + \gamma \int_t^T e^{\beta A_{t,s}} \alpha_s dA_s \right) \middle| \mathcal{F}_t \right] \\ &\leq X_t. \end{aligned} \tag{3.16}$$

In view of the definition of τ_m , we have

$$\begin{aligned} |Y_{t \wedge \tau_m}^{n,k}| &\leq X_{t \wedge \tau_m} \leq m, \\ \|f^{n,k}(\cdot, 0, 0)\|_{\tau_m} &\leq \|\mathbb{I}_{[0, \tau_m]} \alpha\|_{\tau_m} \leq m. \end{aligned} \tag{3.17}$$

Hence $\|f^{n,k}(\cdot, 0, 0)\|$ and $Y^{n,k}$ are uniformly bounded on $[0, \tau_m]$. Secondly, given $(Y^{n,k}, Z^{n,k}, M + N^{n,k})$ which solves $(f^{n,k}, g, \xi^{n,k})$, it is immediate that $(Y_{\cdot \wedge \tau_m}^{n,k}, (Z^{n,k} \cdot M + N^{n,k})_{\cdot \wedge \tau_m})$ solves $(\mathbb{I}_{[0, \tau_m]}(t)f^{n,k}(t, y, z), g, Y_{\tau_m}^{n,k})$. We then use Theorem 3.6 as in Proposition 3.7 to construct a pair $(\tilde{Y}^m, (\tilde{Z}^m \cdot M + \tilde{N}^m))$ which solves $(f, g, \sup_n \inf_k Y_{\tau_m}^{n,k})$, i.e.,

$$\tilde{Y}_t^m = \sup_n \inf_k Y_{\tau_m}^{n,k} + \int_{t \wedge \tau_m}^{\tau_m} (F(s, \tilde{Y}_s^m, \tilde{Z}_s^m) dA_s + g_s \langle \tilde{N}^m \rangle_s) - \int_{t \wedge \tau_m}^{\tau_m} (\tilde{Z}_s^m dM_s + d\tilde{N}_s). \quad (3.18)$$

Moreover, \tilde{Y}^m is the \mathbb{P} -a.s. uniform limit of $Y_{\cdot \wedge \tau_m}^{n,k}$ and $\tilde{Z}^m \cdot M + \tilde{N}^m$ is the \mathcal{M}^2 -limit of $(Z^{n,k} \cdot M + N^{n,k})_{\cdot \wedge \tau_m}$ as k, n go to $+\infty$. Hence

$$\begin{aligned} \tilde{Y}_{\cdot \wedge \tau_m}^{m+1} &= \tilde{Y}_{\cdot \wedge \tau_m}^m \quad \mathbb{P}\text{-a.s.}, \\ \mathbb{I}_{\{t \leq \tau_m\}} \lambda_t \tilde{Z}_t^{m+1} &= \lambda_t \tilde{Z}_t^m dA \otimes d\mathbb{P}\text{-a.e.}, \\ \tilde{N}_{\cdot \wedge \tau_m}^{m+1} &= \tilde{N}_{\cdot \wedge \tau_m}^m \quad \mathbb{P}\text{-a.s.} \end{aligned} \quad (3.19)$$

Define (Y, Z, N) on $[0, T]$ by

$$\begin{aligned} Y_t &:= \mathbb{I}_{\{t \leq \tau_m\}} \tilde{Y}_t^1 + \sum_{m \geq 2} \mathbb{I}_{] \tau_{m-1}, \tau_m]} \tilde{Y}_t^m, \\ Z_t &:= \mathbb{I}_{\{t \leq \tau_m\}} \tilde{Z}_t^1 + \sum_{m \geq 2} \mathbb{I}_{] \tau_{m-1}, \tau_m]} \tilde{Z}_t^m, \\ N_t &:= \mathbb{I}_{\{t \leq \tau_m\}} \tilde{N}_t^1 + \sum_{m \geq 2} \mathbb{I}_{] \tau_{m-1}, \tau_m]} \tilde{N}_t^m. \end{aligned}$$

By (3.19), we have $Y_{\cdot \wedge \tau_m} = \tilde{Y}_{\cdot \wedge \tau_m}^m$, $\mathbb{I}_{\{t \leq \tau_m\}} Z_t = \mathbb{I}_{\{t \leq \tau_m\}} \tilde{Z}_t^m$ and $N_{\cdot \wedge \tau_m} = \tilde{N}_{\cdot \wedge \tau_m}^m$. Hence we can rewrite (3.18) as

$$Y_{t \wedge \tau_m} = \sup_n \inf_k Y_{\tau_m}^{n,k} + \int_{t \wedge \tau_m}^{\tau_m} (f(s, Y_s, Z_s) dA_s + g_s d\langle N \rangle_s) - \int_{t \wedge \tau_m}^{\tau_m} (Z_s dM_s + dN_s).$$

By sending m to $+\infty$, we prove that (Y, Z, N) solves (f, g, ξ) . By (3.16), we have

$$|Y_t| = |\sup_n \inf_k Y_t^{n,k}| \leq \frac{1}{\gamma} \ln \mathbb{E} \left[\exp \left(\gamma e^{\beta A_{t,T}} |\xi| + \gamma \int_t^T e^{\beta A_{t,s}} \alpha_s dA_s \right) \middle| \mathcal{F}_t \right].$$

□

Compared to Mocha and Westray [25], we prove the existence result under rather milder structure conditions. For example, (A.2')(ii) gets rid of linear growth in y and allows g to be any bounded process, which has been seen repeatedly throughout this

chapter. Secondly, in contrast to their work, $dA_t \ll c_A dt$, where c_A is a positive constant, is not needed. Finally, they use a regularization procedure through quadratic BSDEs with bounded data. Hence, more demanding structure conditions are imposed to ensure that the comparison theorem holds. On the contrary, the Lipschitz-quadratic regularization is more direct and essentially merely relies on (A.2') which is the most general assumption to our knowledge. This coincides with Briand and Hu [9] for Brownian framework.

Due to the same reason as in Proposition 3.7, the existence of a maximal or minimal solution is not available.

Remark. Analogously to Hu and Schweizer [19], one may easily extend the existence result to infinite-horizon case. In abstract terms, given exponential moments integrability on $\exp(\beta A_\infty)|\alpha|_\infty$, we regularize through Lipschitz-quadratic BSDEs with increasing horizons and null terminal value. Using a localization procedure and the monotone stability result as in Theorem 3.9, we obtain a solution which solves the infinite-horizon BSDE.

As a result from Lemma 3.8, we derive the estimates for the local martingale part. To save pages we only consider the following extremal case.

Corollary 3.10 (Estimate) *Let (A.2') hold for (f, g, ξ) and $e^{\beta A_T}(|\xi| + |\alpha|_T)$ has exponential moments of all orders. Then any solution (Y, Z, N) verifying (3.13) satisfies: Y has exponential moments of all order and $Z \cdot M + N \in \mathcal{M}^p$ for all $p \geq 1$. More precisely, for all $p > 1$,*

$$\mathbb{E}[e^{p\gamma Y^*}] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}\left[\exp\left(p\gamma e^{\beta A_T}(|\xi| + |\alpha|_T)\right)\right],$$

and for all $p \geq 1$,

$$\mathbb{E}\left[\left(\int_0^T \left(Z_s^\top d\langle M \rangle_s Z_s + d\langle N \rangle_s\right)\right)^{\frac{p}{2}}\right] \leq c \mathbb{E}\left[\exp\left(4p\gamma e^{\beta A_T}(|\xi| + |\alpha|_T)\right)\right],$$

where c is a constant only depending on p, γ .

Proof. The proof is exactly the same as Corollary 4.2, Mocha and Westray [25] and hence omitted. □

Let us turn to the uniqueness result. We modify Mocha and Westray [25] to allow g to be any bounded process rather than merely a constant. A convexity assumption is imposed so as to use θ -technique which proves to be convenient to treat quadratic terms. We start from comparison theorem and then move to uniqueness and stability result. Similar results can be found in Briand and Hu [9] for Brownian setting or Da Lio and Ley [11] from the point of view of PDEs. To this end, the following structure conditions on (f, g, ξ) are needed.

Assumption (A.3) There exist $\beta \geq 0, \gamma > 0$ and an \mathbb{R}^+ -valued Prog-measurable process α such that \mathbb{P} -a.s.

- (i) for any $t \in [0, T]$, $(y, z) \mapsto f(t, y, z)$ is continuous;
- (ii) f is Lipschitz-continuous in y , i.e., for any $(t, z) \in [0, T] \times \mathbb{R}^d$, $y, y' \in \mathbb{R}$,

$$|f(t, y, z) - f(t, y', z)| \leq \beta|y - y'|;$$

- (iii) for any $(t, y) \in [0, T] \times \mathbb{R}$, $z \mapsto f(t, y, z)$ is convex;

- (iv) for any $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$,

$$|f(t, y, z)| \leq \alpha_t + \beta|y| + \frac{\gamma}{2}|\lambda_t z|^2.$$

We start our proof of comparison theorem by observing that (A.3) implies (A.2'). Hence existence is ensured given suitable integrability. Likewise, we keep in mind that \mathbb{P} -a.s. $|g| \leq \frac{\gamma}{2}$.

Theorem 3.11 (Comparison Theorem) *Let $(Y, Z \cdot M + N), (Y', Z' \cdot M + N') \in \mathcal{S} \times \mathcal{M}$ be solutions to $(f, g, \xi), (f', g', \xi')$, respectively, and $Y^*, (Y')^*, |\alpha|_T$ have exponential moments of all orders. If \mathbb{P} -a.s. for any $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$, $f(t, y, z) \leq f'(t, y, z)$, $g_t \leq g'_t$, $g'_t \geq 0$, $\xi \leq \xi'$ and (f, g, ξ) verifies (A.3), then \mathbb{P} -a.s. $Y \leq Y'$.*

Proof. We introduce the notations used throughout the proof. For any $\theta \in (0, 1)$, define

$$\begin{aligned} \delta f_t &:= f(t, Y'_t, Z'_t) - f'(t, Y'_t, Z'_t), \\ \delta_\theta Y &:= Y - \theta Y', \\ \delta Y &:= Y - Y', \end{aligned}$$

and $\delta_\theta Z, \delta Z, \delta_\theta N, \delta N$, etc. analogously. Moreover, define

$$\rho_t := \mathbb{I}_{\{\delta_\theta Y_t \neq 0\}} \frac{f(t, Y_t, Z_t) - f(t, \theta Y'_t, Z_t)}{\delta_\theta Y_t}.$$

By (A.3)(ii), ρ is bounded by β for any $\theta \in (0, 1)$. Hence $|\rho|_T \leq \beta \|A\|$. By Itô's formula,

$$\begin{aligned} e^{|\rho|_t} \delta_\theta Y_t &= e^{|\rho|_T} \delta_\theta Y_T + \int_t^T e^{|\rho|_s} F_s^\theta dA_s + \int_t^T e^{|\rho|_s} (g_s d\langle N \rangle_s - \theta g'_s d\langle N' \rangle_s) \\ &\quad - \int_t^T e^{|\rho|_s} (\delta_\theta Z_s dM_s + d\delta_\theta N_s), \end{aligned}$$

where

$$\begin{aligned} F_s^\theta &= f(s, Y_s, Z_s) - \theta f'(s, Y'_s, Z'_s) - \rho_s \delta_\theta Y_s, \\ &= \theta \delta f_s + (f(s, Y_s, Z_s) - f(s, Y'_s, Z'_s)) + (f(s, Y'_s, Z_s) - \theta f(s, Y'_s, Z'_s)) - \rho_s \delta_\theta Y_s. \end{aligned} \quad (3.20)$$

We then use (A.3)(ii)(iii) to deduce that

$$\begin{aligned} f(s, Y_s, Z_s) - f(t, Y'_s, Z'_s) &= f(s, Y_s, Z_s) - f(s, \theta Y'_s, Z_s) + f(s, \theta Y'_s, Z_s) - f(s, Y'_s, Z_s) \\ &= \rho_s \delta_\theta Y_s + f(t, \theta Y'_s, Z_s) - f(s, Y'_s, Z_s) \\ &\leq \rho_s \delta_\theta Y_s + (1 - \theta) \beta |Y'_s|, \\ f(s, Y'_s, Z_s) - \theta f(s, Y'_s, Z'_s) &= f(s, Y'_s, \theta Z'_s + (1 - \theta) \frac{\delta_\theta Z_s}{1 - \theta}) - \theta f(t, Y'_s, Z'_s) \\ &\leq (1 - \theta) f(s, Y'_s, \frac{\delta_\theta Z_s}{1 - \theta}) \\ &\leq (1 - \theta) \alpha_s + (1 - \theta) \beta |Y'_s| + \frac{\gamma}{2(1 - \theta)} |\lambda_s \delta_\theta Z_s|^2. \end{aligned}$$

We also note that \mathbb{P} -a.s. $\delta f_s \leq 0$. Hence plugging these inequalities into (3.20) gives

$$F_s^\theta \leq (1 - \theta) (\alpha_s + 2\beta |Y'_s|) + \frac{\gamma}{2(1 - \theta)} |\lambda_s \delta_\theta Z_s|^2. \quad (3.21)$$

We then perform an exponential transform to eliminate both quadratic terms. Set

$$\begin{aligned} c &:= \frac{\gamma e^{\beta \|A\|}}{1 - \theta}, \\ P_t &:= \exp \left(c e^{|\rho|_t} \delta_\theta Y_t \right). \end{aligned}$$

By Itô's formula,

$$\begin{aligned} P_t &= P_T + \int_t^T c P_s e^{|\rho|_s} \left(F_s^\theta - \frac{c e^{|\rho|_s}}{2} |\delta_\theta Z_s|^2 \right) dA_s \\ &\quad + \int_t^T c P_s e^{|\rho|_s} \left(g_s d\langle N \rangle_s - \theta g'_s d\langle N' \rangle_s - \frac{c e^{|\rho|_s}}{2} d\langle \delta_\theta N \rangle_s \right) \\ &\quad - \int_t^T c P_s e^{|\rho|_s} (\delta_\theta Z_s dM_s + d\delta_\theta N_s). \end{aligned}$$

For notational convenience, we define

$$\begin{aligned} G_t &:= c P_t e^{|\rho|_t} \left(F_t^\theta - \frac{c e^{|\rho|_t}}{2} |Z_t^\theta|^2 \right), \\ H_t &:= \int_0^t c P_s e^{|\rho|_s} \left(g_s d\langle N \rangle_s - \theta g'_s d\langle N' \rangle_s - \frac{c e^{|\rho|_s}}{2} d\langle N^\theta \rangle_s \right). \end{aligned}$$

By (3.21), we have

$$G_t = cP_t e^{|\rho|t} \left((1 - \theta)(\alpha_t + 2\beta|Y'_t|) \right) \leq P_t J_t,$$

where

$$J_t := \gamma e^{2\beta\|A\|} (\alpha_t + 2\beta|Y'_t|).$$

We claim that H can also be eliminated. Indeed,

$$\begin{aligned} d\langle \delta_\theta N \rangle &= d\langle N \rangle + \theta^2 d\langle N' \rangle - 2\theta d\langle N, N' \rangle \\ &\gg d\langle N \rangle + \theta^2 d\langle N' \rangle - \theta d\langle N \rangle - \theta d\langle N' \rangle \\ &= (1 - \theta)(d\langle N \rangle - \theta d\langle N' \rangle) \\ &= (1 - \theta)d\delta_\theta \langle N \rangle. \end{aligned}$$

We then come back to H and use this inequality to deduce that

$$\begin{aligned} g_t d\langle N \rangle_t - \theta g'_t d\langle N' \rangle_t - \frac{ce^{|\rho|t}}{2} d\langle \delta_\theta N \rangle_t &= g_t^+ d\langle N \rangle_t - g_t^- d\langle N \rangle_t - \theta g'_t d\langle N' \rangle_t - \frac{ce^{|\rho|t}}{2} d\langle \delta_\theta N \rangle_t \\ &\ll g_t^+ d\delta_\theta \langle N \rangle_t + \theta(g_t^+ - g'_t) d\langle N' \rangle_t - \frac{ce^{|\rho|t}}{2} d\langle \delta_\theta N \rangle_t \\ &\ll g_t^+ d\delta_\theta \langle N \rangle_t - \frac{\gamma}{2(1 - \theta)} d\langle \delta_\theta N \rangle_t \\ &\ll 0, \end{aligned}$$

due to $g^+ \leq g'$ and $g \leq \frac{\gamma}{2}$. Hence $dH_t \ll 0$. To eliminate G , we set $D_t := \exp(|J|_t)$. By Itô's formula,

$$d(D_t P_t) = D_t \left((P_t J_t - G_t) dA_t - dH_t + cP_t e^{|\rho|t} (\delta_\theta Z_t dM_t + d\delta_\theta N_t) \right).$$

But previous results show that $(P_t J_t - G_t) dA_t - dH_t \gg 0$. Hence DP is a local submartingale. Thanks to the exponential moments integrability on $|\alpha|_T$ and $(Y')^*$ (and hence $|J|_T$), we use a localization procedure and the same arguments in Proposition 2.5 to deduce that

$$P_t \leq \mathbb{E} \left[\exp \left(\int_t^T J_s dA_s \right) P_T \middle| \mathcal{F}_t \right]. \quad (3.22)$$

We come back to the definition of P_T and observe that

$$\begin{aligned} \delta_\theta \xi &\leq (1 - \theta)|\xi| + \theta \delta \xi \\ &\leq (1 - \theta)|\xi|. \end{aligned}$$

Hence (3.22) gives

$$\begin{aligned} \exp\left(\frac{\gamma e^{\beta\|A\|+|\rho|_t}}{1-\theta}\delta_\theta Y_t\right) &\leq \mathbb{E}\left[\exp\left(\int_t^T J_s dA_s\right) \exp\left(ce^{|\rho|_T}\delta_\theta \xi\right)\middle|\mathcal{F}_t\right] \\ &\leq \mathbb{E}\left[\exp\left(\int_t^T J_s dA_s\right) \exp\left(\gamma e^{2\beta\|A\|}|\xi|\right)\middle|\mathcal{F}_t\right]. \end{aligned}$$

Hence

$$\delta_\theta Y_t \leq \frac{1-\theta}{\gamma} \ln \mathbb{E}\left[\exp\left(\gamma e^{2\beta\|A\|}\left(|\xi| + \int_t^T (\alpha_s + 2\beta|Y'_s|)dA_s\right)\right)\middle|\mathcal{F}_t\right].$$

Therefore we obtain \mathbb{P} -a.s. $Y_t \leq Y'_t$, by sending θ to 1. By the continuity of Y and Y' , we also have \mathbb{P} -a.s. $Y \leq Y'$.

□

As a byproduct, we can prove the existence of a unique solution given (A.3).

Corollary 3.12 (Uniqueness) *If (f, g, ξ) satisfies (A.3), \mathbb{P} -a.s. $g \geq 0$ and $|\xi|, |\alpha|_T$ have exponential moments of all orders, then there exists a unique solution (Y, Z, N) to (f, g, ξ) such that Y^* has exponential moments of all order and $(Z \cdot M + N) \in \mathcal{M}^p$ for all $p \geq 1$.*

Proof. The existence of a unique solution in the above sense is immediate from Theorem 3.9 (existence), Theorem 3.11 (comparison theorem) and Corollary 3.10 (estimate).

□

Remark. There are spaces to sharpen the uniqueness. The convexity in z motivates one to replace (A.3)(iv) by

$$-\underline{\alpha}_t - \beta|y| - \kappa|\lambda_t z| \leq f(t, y, z) \leq \bar{\alpha}_t + \beta|y| + \frac{\gamma}{2}|\lambda_t z|^2.$$

Secondly, in view of Delbaen et al [12], we may prove uniqueness given weaker integrability, by characterizing the solution as the value process of a stochastic control problem.

It turns out that a stability result also holds given convexity condition. The proof is a modification of Theorem 3.11 (comparison theorem). We set $\mathbb{N}^0 := \mathbb{N}^+ \cup \{0\}$.

Proposition 3.13 (Stability) *Let $(f^n, g^n, \xi^n)_{n \in \mathbb{N}^0}$ with $g^n \geq 0$ \mathbb{P} -a.s. satisfy (A.3) associated with $(\alpha^n, \beta, \gamma, \varphi)$, and (Y^n, Z^n, N^n) be their unique solutions in the sense*

of Corollary 3.12, respectively. If $\xi^n - \xi^0 \rightarrow 0$, $\int_0^T |f^n - f^0|(s, Y_s^0, Z_s^0) dA_s \rightarrow 0$ in probability, \mathbb{P} -a.s. $g^n - g^0 \rightarrow 0$ as n goes to $+\infty$ and for each $p > 0$,

$$\begin{aligned} \sup_{n \in \mathbb{N}^0} \mathbb{E} \left[\exp \left(p(|\xi^n| + |\alpha^n|_T) \right) \right] &< +\infty, \\ \sup_{n \in \mathbb{N}^0} |g^n| &\leq \frac{\gamma}{2} \quad \mathbb{P}\text{-a.s.} \end{aligned} \quad (3.23)$$

Then for each $p \geq 1$,

$$\begin{aligned} \lim_n \mathbb{E} \left[\exp \left(p|Y^n - Y^0|^* \right) \right] &= 1, \\ \lim_n \mathbb{E} \left[\left(\int_0^T \left((Z_s^n - Z_s^0)^\top d\langle M \rangle_s (Z_s^n - Z_s^0) + d\langle N^n - N^0 \rangle_s \right) \right)^{\frac{p}{2}} \right] &= 0. \end{aligned}$$

Proof. By Corollary 3.10 (estimate), for any $p \geq 1$,

$$\sup_{n \in \mathbb{N}^0} \mathbb{E} \left[\exp \left(p(Y^n)^* \right) + \left(\int_0^T \left((Z_s^n)^\top d\langle M \rangle_s Z_s^n + d\langle N^n \rangle_s \right) \right)^{\frac{p}{2}} \right] < +\infty. \quad (3.24)$$

Hence the sequence of random variables

$$\exp \left(p|Y^n - Y^0|^* \right) + \left(\int_0^T \left((Z_s^n - Z_s^0)^\top d\langle M \rangle_s (Z_s^n - Z_s^0) + d\langle N^n - N^0 \rangle_s \right) \right)^{\frac{p}{2}}$$

is uniformly integrable. Due to Vitali convergence, it is hence sufficient to prove that

$$|Y^n - Y|^* + \int_0^T \left((Z_s^n - Z_s^0)^\top d\langle M \rangle_s (Z_s^n - Z_s^0) + d\langle N^n - N \rangle_s \right) \rightarrow 0$$

in probability as n goes to $+\infty$.

(i). We prove *u.c.p* convergence of $Y^n - Y^0$. To this end we use θ -technique in the spirit of Theorem 3.11 (comparison theorem). For any $\theta \in (0, 1)$, define

$$\begin{aligned} \delta f_t^n &:= f^0(t, Y_t^0, Z_t^0) - f^n(t, Y_t^0, Z_t^0), \\ \delta g^n &:= g^0 - g^n, \\ \delta_\theta Y^n &:= Y^0 - \theta Y^n, \end{aligned}$$

and $\delta_\theta Z^n, \delta_\theta N^n, \delta_\theta \langle N \rangle^n$, etc. analogously. Further, set

$$\begin{aligned} \rho_t &:= \mathbb{I}_{\{Y_t^0 - Y_t^n \neq 0\}} \frac{f^n(t, Y_t^0, Z_t^n) - f^n(t, Y_t^n, Z_t^n)}{Y_t^0 - Y_t^n}, \\ c &:= \frac{\gamma e^{\beta \|A\|}}{1 - \theta}, \\ P_t^n &:= \exp \left(c e^{|\rho|_t} \delta_\theta Y_t^n \right), \\ J_t^n &:= \gamma e^{2\beta \|A\|} (\alpha_t^n + 2\beta |Y_t^0|), \\ D_t^n &:= \exp \left(\int_0^t J_s^n dA_s \right). \end{aligned}$$

Obviously ρ is bounded by β due to (A.3)(i). The θ -difference implies that

$$\begin{aligned} & f^0(t, Y_t^0, Z_t^0) - \theta f^n(t, Y_t^n, Z_t^n) \\ &= \delta f_t^n + (\theta f^n(t, Y_t^0, Z_t^n) - \theta f^n(t, Y_t^n, Z_t^n)) + (f^n(t, Y_t^0, Z_t^0) - \theta f^n(t, Y_t^0, Z_t^n)). \end{aligned} \quad (3.25)$$

By (A.3)(i)(ii),

$$\begin{aligned} \theta f^n(t, Y_t^0, Z_t^n) - \theta f^n(t, Y_t^n, Z_t^n) &= \theta \rho_t (Y_t^0 - Y_t^n) \\ &= \rho_t (\theta Y_t^0 - Y_t^0 + Y_t^0 - \theta Y_t^n) \\ &\leq (1 - \theta) \beta |Y_t^0| + \rho_t \delta_\theta Y_t^n, \\ f^n(t, Y_t^0, Z_t^0) - \theta f^n(t, Y_t^0, Z_t^n) &\leq (1 - \theta) \alpha_t^n + (1 - \theta) \beta |Y_t^0| + \frac{\gamma}{2(1 - \theta)} |\delta_\theta Z_t^n|^2. \end{aligned}$$

Hence (3.25) gives

$$f^0(t, Y_t^0, Z_t^0) - \theta f^n(t, Y_t^n, Z_t^n) - \rho_t \delta_\theta Y_t^n \leq \delta f_t^n + (1 - \theta) (\alpha_t^n + 2\beta |Y_t^0|) + \frac{\gamma}{2(1 - \theta)} |\delta_\theta Z_t^n|^2. \quad (3.26)$$

To analyze the quadratic term concerning N^0 and N^n , we deduce by the same arguments as in Theorem 3.11 that

$$\begin{aligned} g_t^0 d\langle N^0 \rangle_t - \theta g_t^n d\langle N^n \rangle_t - \frac{ce^{|\rho|t}}{2} d\langle \delta_\theta N \rangle_t &= \delta g_t^n d\langle N^0 \rangle_t + g_t^n d\delta_\theta \langle N \rangle_t^n - \frac{ce^{|\rho|t}}{2} d\langle \delta_\theta N^n \rangle_t \\ &\ll g_t^n \left(d\delta_\theta \langle N \rangle_t^n - \frac{1}{1 - \theta} d\langle \delta_\theta N^n \rangle_t \right) + \delta g_t^n d\langle N^0 \rangle_t \\ &\ll \delta g_t^n d\langle N^0 \rangle_t. \end{aligned} \quad (3.27)$$

Given (3.26) and (3.27), we use an exponential transform which is analogous to that in Theorem 3.11. This gives

$$P_t^n \leq D_t^n P_t^n \leq \mathbb{E} \left[D_T^n P_T^n + \frac{\gamma e^{2\beta \|A\|}}{1 - \theta} \int_t^T D_s^n P_s^n (|\delta f_s^n| dA_s + |\delta g_s^n| d\langle N^0 \rangle_s) \middle| \mathcal{F}_t \right].$$

Using $\log x \leq x$ and $Y^0 - Y^n \leq (1 - \theta) |Y^n| + \delta_\theta Y^n$, we deduce that

$$Y_t^0 - Y_t^n \leq (1 - \theta) |Y_t^n| + \frac{1 - \theta}{\gamma} \mathbb{E} \left[D_T^n P_T^n + \frac{\gamma e^{2\beta \|A\|}}{1 - \theta} \int_t^T D_s^n P_s^n (|\delta f_s^n| dA_s + |\delta g_s^n| d\langle N^0 \rangle_s) \middle| \mathcal{F}_t \right].$$

Set

$$\begin{aligned} \Lambda^n(\theta) &:= \exp \left(\frac{\gamma e^{2\beta \|A\|}}{1 - \theta} ((Y^0)^* + (Y^n)^*) \right) \geq P_t^n, \\ \Xi^n(\theta) &:= \exp \left(\frac{\gamma e^{2\beta \|A\|}}{1 - \theta} (|\xi^0 - \theta \xi^n| \vee |\xi^n - \theta \xi^0|) \right) \geq P_T^n. \end{aligned}$$

We then have

$$Y_t^0 - Y_t^n \leq (1-\theta)|Y_t^n| + \frac{1-\theta}{\gamma} \mathbb{E} \left[D_T^n \Xi^n(\theta) + \frac{\gamma e^{2\beta\|A\|}}{1-\theta} D_T^n \Lambda^n(\theta) \int_t^T (|\delta f_s^n| dA_s + |\delta g_s^n| d\langle N^0 \rangle_s) \middle| \mathcal{F}_t \right].$$

Now we use (A.3)(ii)(iii) to f^n and proceed analogously to Theorem 3.11. This gives

$$Y_t^n - Y_t^0 \leq (1-\theta)|Y_t^0| + \frac{1-\theta}{\gamma} \mathbb{E} \left[D_T^n \Xi^n(\theta) + \frac{\gamma e^{2\beta\|A\|}}{1-\theta} D_T^n \Lambda^n(\theta) \int_t^T (|\delta f_s^n| dA_s + |\delta g_s^n| d\langle N^0 \rangle_s) \middle| \mathcal{F}_t \right].$$

Though looking symmetric, the two inequalities come from slightly different treatments for the θ -difference. The two estimates give

$$\begin{aligned} |Y_t^n - Y_t^0| &\leq \underbrace{(1-\theta)(|Y_t^0| + |Y_t^n|)}_{X_t^1} + \underbrace{\frac{1-\theta}{\gamma} \mathbb{E} \left[D_T^n \Xi^n(\theta) \middle| \mathcal{F}_t \right]}_{X_t^2} \\ &\quad + \underbrace{e^{2\beta\|A\|} \mathbb{E} \left[D_T^n \Lambda^n(\theta) \int_0^T (|\delta f_s^n| dA_s + |\delta g_s^n| d\langle N^0 \rangle_s) \middle| \mathcal{F}_t \right]}_{X_t^3}. \end{aligned}$$

We then prove *u.c.p* convergence of $Y^n - Y^0$. For any $\epsilon > 0$,

$$\mathbb{P}(|Y^n - Y^0|^* \geq \epsilon) \leq \mathbb{P}((X^1)^* \geq \frac{\epsilon}{3}) + \mathbb{P}((X^2)^* \geq \frac{\epsilon}{3}) + \mathbb{P}((X^3)^* \geq \frac{\epsilon}{3}). \quad (3.28)$$

We aim at showing that each term on the right-hand side of (3.28) converges to 0 if we send n to $+\infty$ first and then θ to 1. To this end, we give some useful estimates. By Chebyshev's inequality,

$$\mathbb{P}((X^1)^* \geq \frac{\epsilon}{3}) \leq \frac{3(1-\theta)}{\epsilon} \mathbb{E}[(Y^0)^* + (Y^n)^*],$$

where $\mathbb{E}[(Y^0)^* + (Y^n)^*]$ is uniformly bounded. Secondly, Doob's inequality yields

$$\mathbb{P}((X^2)^* \geq \frac{\epsilon}{3}) \leq \frac{3(1-\theta)\gamma}{\epsilon} \mathbb{E}[D_T^n \Xi_T^n]. \quad (3.29)$$

Moreover, by Vitali convergence, the right-hand side of (3.29) satisfies

$$\begin{aligned} \limsup_n \mathbb{E}[D_T^n \Xi_T^n] &\leq \sup_n \mathbb{E}[(D^n)^2]^{\frac{1}{2}} \cdot \limsup_n \mathbb{E}[(\Xi^n)^2]^{\frac{1}{2}} \\ &\leq \sup_n \mathbb{E}[(D^n)^2]^{\frac{1}{2}} \cdot \mathbb{E} \left[\exp \left(2\gamma e^{2\beta\|A\|} |\xi^0| \right) \right]^{\frac{1}{2}} \\ &< +\infty. \end{aligned}$$

Hence, the first term and the second term on the right-hand side of (3.28) converge to 0 as n goes to $+\infty$ and θ goes to 1. Finally, we claim that the third term on the right-hand side of (3.28) also converges. Indeed, Doob's inequality and Hölder's inequality give

$$\begin{aligned} \mathbb{P}\left((X^3)^* \geq \frac{\epsilon}{3}\right) &\leq \frac{3e^{2\beta\|A\|}}{\epsilon} \mathbb{E}\left[D_T^n \Lambda^n(\theta) \int_t^T (|\delta f_s^n| dA_s + |\delta g_s^n| d\langle N^0 \rangle_s)\right] \\ &\leq \frac{3e^{2\beta\|A\|}}{\epsilon} \mathbb{E}\left[(D_T^n \Lambda^n(\theta))^2\right]^{\frac{1}{2}} \mathbb{E}\left[\left(\int_0^T (|\delta f_s^n| dA_s + |\delta g_s^n| d\langle N^0 \rangle_s)\right)^2\right]^{\frac{1}{2}}. \end{aligned} \quad (3.30)$$

Note that

$$\int_0^T (|\delta f_s^n| dA_s + |\delta g_s^n| d\langle N^0 \rangle_s) \leq |\alpha|_T + |\alpha^n|_T + 2\|A\|(Y^0)^* + \gamma\langle Z^0 \cdot M + N^0 \rangle_T.$$

Hence the left-hand side of this inequality has finite moments of all orders by Corollary 3.10. Therefore, the left-hand side of (3.30) converges to 0 as n goes to $+\infty$ due to Vitali convergence.

Finally, collecting these convergence results for each term in (3.28) gives the convergence of $Y^n - Y^0$.

(ii). It remains to prove convergence of the martingale parts. By Itô's formula,

$$\begin{aligned} &\mathbb{E}\left[\int_0^T \left((Z_s^n - Z_s^0)^\top d\langle M \rangle_s (Z_s^n - Z_s^0) + d\langle N^n - N^0 \rangle_s\right)\right] \\ &\leq \mathbb{E}[|\xi^n - \xi^0|^2] + 2\mathbb{E}\left[|Y^n - Y^0|^* \int_0^T |F^n(s, Y_s^n, Z_s^n) - F^0(s, Y_s^0, Z_s^0)| dA_s\right] \\ &\quad + 2\mathbb{E}\left[|Y^n - Y^0|^* \left|\int_0^T (g_s^n d\langle N^n \rangle_s - g_s^0 d\langle N^0 \rangle_s)\right|\right], \end{aligned}$$

As before, we conclude by Vitali convergence.

□

3.5 Change of Measure

We show that given exponential moments integrability, the martingale part $Z \cdot M + N$, though not BMO, defines an equivalent change of measure, i.e., its stochastic exponential is a strictly positive martingale. We don't require convexity which ensures uniqueness. But to derive the estimate for $\int_0^T f(s, Y_s, Z_s) dA_s$, we use (A.2'') where f is of linear growth in y . We keep assuming that \mathbb{P} -a.s. $|g| \leq \frac{\gamma}{2}$. The following result comes from Mocha and Westray [25].

Theorem 3.14 (Change of Measure) *If (f, g, ξ) satisfies (A.2'') and $\xi, |\alpha|_T$ have exponential moments of all orders, then for any solution (Y, Z, N) such that Y has exponential moments of all orders and any $|q| > \frac{\gamma}{2}$, $\mathcal{E}(q(Z \cdot M + N))$ is a continuous martingale.*

Proof. We start by recalling Lemma 1.6. and Lemma 1.7., Kazamaki [21]: if \widetilde{M} is a martingale such that

$$\sup_{\tau \in \mathcal{T}} \mathbb{E} \left[\exp \left(\eta \widetilde{M}_\tau + \left(\frac{1}{2} - \eta \right) \langle \widetilde{M} \rangle_\tau \right) \right] < +\infty, \quad (3.31)$$

for $\eta \neq 1$, then $\mathcal{E}(\eta \widetilde{M})$ is a martingale. Moreover, if (3.31) holds for some $\eta^* > 1$ then it holds for any $\eta \in (1, \eta^*)$.

By Lemma 3.10 (estimate), $Z \cdot M + N$ is a continuous martingale. First of all, we apply the above criterion to $\widetilde{M} := \tilde{q}(Z \cdot M + N)$ for some fixed $|\tilde{q}| > \frac{\gamma}{2}$. Define $\Lambda_t(\eta)$ such that

$$\ln \Lambda_t(\eta) := \tilde{q}\eta((Z \cdot M)_t + N_t) + \tilde{q}^2 \left(\frac{1}{2} - \eta \right) \langle Z \cdot M + N \rangle_t.$$

From the BSDE (3.1) and (A.2''), we obtain, for any $\tau \in \mathcal{T}$,

$$\begin{aligned} \ln \Lambda_\tau(\eta) &= \tilde{q}\eta \left(Y_t - Y_0 + \int_0^t (f(s, Y_s, Z_s) dA_s + g_s d\langle N \rangle_s) \right) + \tilde{q}^2 \left(\frac{1}{2} - \eta \right) \langle Z \cdot M + N \rangle_t \\ &\leq (2 + \beta \|A\|) |\tilde{q}| \eta Y^* + |\tilde{q}| \eta |\alpha|_T + |\tilde{q}| \eta \left(\frac{\gamma}{2} + \frac{|\tilde{q}|}{\eta} \left(\frac{1}{2} - \eta \right) \right) \langle Z \cdot M + N \rangle_T. \end{aligned} \quad (3.32)$$

Note that

$$\frac{\gamma}{2} + \frac{|\tilde{q}|}{\eta} \left(\frac{1}{2} - \eta \right) \leq 0 \iff \eta \geq \frac{|\tilde{q}|}{2|\tilde{q}| - \gamma} =: q_0 \left(> \frac{1}{2} \right).$$

Hence for any $\eta \geq q_0$, (3.32) gives

$$\Lambda_\tau(\eta) \leq \exp \left(|\tilde{q}| \eta (2 + \beta) Y_* + |\tilde{q}| \eta |\alpha|_T \right).$$

By exponential moments integrability, we have

$$\sup_{\tau \in \mathcal{T}} \mathbb{E} [\Lambda_\tau(\eta)] < +\infty.$$

It then follows from the first statement of the criterion that $\mathcal{E}(\tilde{q}\eta(Z \cdot M + N))$ is a martingale for all $\eta \in [q_0, \infty) \setminus \{1\}$. The second statement ensures that it is a martingale for any $\eta > 1$. For any $|q| > \frac{\gamma}{2}$, we set $|\tilde{q}| \in (\frac{\gamma}{2}, |q|)$, $\eta := \frac{q}{\tilde{q}} > 1$, and apply the result above to conclude that $\mathcal{E}(q(Z \cdot M + N))$ is a martingale.

□

Chapter 4

Quadratic Semimartingales with Applications to Quadratic BSDEs

4.1 Preliminaries

This chapter is a survey of the monotone stability result for quadratic BSDEs studied by Barrieu and El Karoui [4]. Roughly speaking, it comes from the stability of quadratic semimartingales which are processes characterizing the solutions to BSDEs. We fix the time horizon $T > 0$, and work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ satisfying the usual conditions of right-continuity and \mathbb{P} -completeness. We also assume that \mathcal{F}_0 is the \mathbb{P} -completion of the trivial σ -algebra. Any measurability will refer to the filtration $(\mathcal{F}_t)_{t \in [0, T]}$. In particular, Prog denotes the progressive σ -algebra on $\Omega \times [0, T]$. We assume the filtration is *continuous*, in the sense that all local martingales have \mathbb{P} -a.s. continuous sample paths. As mentioned in the introduction, we exclusively study quadratic semimartingales and BSDEs which are \mathbb{R} -valued.

Now we introduce all required notations for this chapter. \ll stands for the strong order of nondecreasing processes, stating that the difference is nondecreasing. For any random variable or process Y , we say Y has some property if this is true except on a \mathbb{P} -null subset of Ω . Hence we omit “ \mathbb{P} -a.s.” in situations without ambiguity. For any random variable X , define $\|X\|_\infty$ to be its essential supremum. For any càdlàg process Y , set $Y_{s,t} := Y_t - Y_s$ and $Y^* := \sup_{t \in [0, T]} |Y_t|$; we denote its total variation process by $|Y|$. \mathcal{T} stands for the set of all stopping times valued in $[0, T]$ and \mathcal{S} denotes the space of continuous adapted processes. For later use we specify the following spaces under \mathbb{P} .

- \mathcal{S}^∞ : the space of bounded process $Y \in \mathcal{S}$ with $\|Y\| := \|Y^*\|_\infty < +\infty$;
- $\mathcal{S}^p(p \geq 1)$: the set of $Y \in \mathcal{S}$ with $Y^* \in \mathbb{L}^p$;
- \mathcal{M} : the space of \mathbb{R} -valued continuous local martingale;

- $\mathcal{M}^p(p \geq 1)$: the set of $M \in \mathcal{M}$ with

$$\|M\|_{\mathcal{M}^p} := \left(\mathbb{E}[(M^*)^p] \right)^{\frac{1}{p}} < +\infty;$$

\mathcal{M}^p is a Banach space.

Finally, for any local martingale M , we call $\{\sigma_n\}_{n \in \mathbb{N}^+} \subset \mathcal{T}$ a *localizing sequence* if σ_n increases stationarily to T as n goes to $+\infty$ and $M_{\cdot \wedge \sigma_n}$ is a martingale for any $n \in \mathbb{N}^+$.

4.2 Quadratic Semimartingales

In this section, we give the notion and characterizations of quadratic semimartingales.

$\mathcal{Q}(\Lambda, C)$ Semimartingale. Let Y be a continuous semimartingale with canonical decomposition $Y = Y_0 - A + M$, where A is a continuous adapted process of finite variation and M is a continuous local martingale with quadratic variation $\langle M \rangle$. Moreover, let Λ and C be fixed continuous adapted processes of finite variation. We call Y a $\mathcal{Q}(\Lambda, C)$ semimartingale if structure condition $\mathcal{Q}(\Lambda, C)$ holds

$$d|A| \ll d\Lambda + |Y|dC + \frac{1}{2}d\langle M \rangle.$$

When there is no ambiguity, Y is also called a \mathcal{Q} semimartingale or quadratic semimartingale. Obviously, Y is a $\mathcal{Q}(\Lambda, C)$ semimartingale if and only if $-Y$ is a $\mathcal{Q}(\Lambda, C)$ semimartingale. Throughout our study, Λ and C exclusively denote continuous nondecreasing adapted processes in the above form. For any optional process Y , we define

$$D^{\Lambda, C}(Y) := \Lambda + \int_0^\cdot |Y_s|dC_s.$$

We are about to introduce the optional strong submartingales and their decomposition which is crucial to the characterizations of quadratic semimartingales. We present a general definition which doesn't require the filtration to be continuous.

Optional Strong Submartingale. An optional process Y is a strong submartingale if for any $\tau, \sigma \in \mathcal{T}$ with $\tau \leq \sigma$, $\mathbb{E}[Y_\sigma | \mathcal{F}_\tau] \geq Y_\tau$ and Y_σ is integrable.

By Theorem 4, Appendix I, Dellacherie and Meyer [13], every optional strong submartingale is indistinguishable from a $\text{l\`a}d\text{l\`a}g$ process. Hence we assume without loss of generality that all optional strong submartingales are $\text{l\`a}d\text{l\`a}g$. We can also define a (local) optional strong submartingale (respectively supermartingale, martingale) in an obvious way. Though not $\text{c\`a}d\text{l\`a}g$ in general, a local optional strong submartingale also has a decomposition of Doob-Meyer's type called *Mertens decomposition*; see Appendix I, Dellacherie and Meyer [13]. More precisely, if Y is a local optional strong submartingale, then it admits a unique decomposition $Y = Y_0 + A + M$, where A is a nondecreasing

predictable process (which is in general only làdlàg) and M is a local martingale. If Y is càdlàg, Mertens decomposition coincides with Doob-Meyer decomposition.

We then introduce the following process by using the continuity of the filtration.

\mathcal{Q} Submartingale. We call a semimartingale Y a \mathcal{Q} submartingale if $Y = Y_0 - A + M$, where M is a continuous local martingale such that $-A + \frac{1}{2}\langle M \rangle$ is a nondecreasing predictable process.

By Itô's formula, Y is a \mathcal{Q} submartingale if and only if $e^Y = e^{Y_0 - A + \frac{1}{2}\langle M \rangle} \mathcal{E}(M)$ is a local optional submartingale.

For any optional process Y , we define the following optional processes

$$\begin{aligned} X^{\Lambda, C}(Y) &:= Y + \Lambda + \int_0^\cdot |Y_s| dC_s = Y + D^{\Lambda, C}(Y), \\ U^{\Lambda, C}(e^Y) &:= e^Y + \int_0^\cdot e^{Y_s} d\Lambda_s + \int_0^\cdot e^{Y_s} |Y_s| dC_s = e^Y + \int_0^\cdot e^{Y_s} dD_s^{\Lambda, C}(Y). \end{aligned}$$

If Λ and C are fixed, we set $X(Y) := X^{\Lambda, C}(Y)$, $U(e^Y) := U^{\Lambda, C}(e^Y)$ and $D := D^{\Lambda, C}(Y) = D^{\Lambda, C}(-Y)$ when there is no ambiguity. This notation also applies to other processes.

With the above notions and properties, we prove equivalent characterizations of quadratic semimartingales.

Theorem 4.1 (Equivalent Characterizations) *Y is a $\mathcal{Q}(\Lambda, C)$ semimartingale if and only if both $X(Y)$ (respectively $U(Y)$) and $X(-Y)$ (respectively $U(-Y)$) are \mathcal{Q} submartingales (respectively local optional strong submartingales).*

Proof. (i). \implies . Suppose Y has canonical decomposition $Y = Y_0 - A + M$. Hence,

$$X(Y) = Y_0 - A + D + M.$$

By the structure condition $\mathcal{Q}(\Lambda, C)$,

$$-dA \gg -d|A| \gg -dD - \frac{1}{2}d\langle M \rangle.$$

This implies that $-dA + dD + \frac{1}{2}d\langle M \rangle \gg 0$. Hence by definition $X(Y)$ is a \mathcal{Q} submartingale. $X(-Y)$ being also a \mathcal{Q} submartingale is immediate since $-Y$ is a $\mathcal{Q}(\Lambda, C)$ semimartingale.

(ii). \impliedby . Suppose $X(Y)$ and $X(-Y)$ admit the following decomposition

$$\begin{aligned} X(Y) &= Y_0 - \overline{A} + \overline{M}, \\ X(-Y) &= -Y_0 - \underline{A} + \underline{M}, \end{aligned}$$

where $-\overline{A} + \frac{1}{2}\langle \overline{M} \rangle$ and $-\underline{A} + \frac{1}{2}\langle \underline{M} \rangle$ are nondecreasing and predictable. Hence $-\Delta \overline{A}$ and $-\Delta \underline{A}$ are nonnegative. Moreover, the process

$$2D = X(Y) + X(-Y) = -\overline{A} - \underline{A} + \overline{M} + \underline{M}. \quad (4.1)$$

is of finite variation. Therefore $\overline{M} + \underline{M} = 0$ and $\langle \overline{M} \rangle = \langle \underline{M} \rangle$. On the other hand, the continuity of D implies that $\Delta(-\overline{A} - \underline{A}) = 0$. A combination of this fact and $-\Delta\overline{A}, -\Delta\underline{A} \geq 0$ thus shows that $\Delta\overline{A} = 0$ and $\Delta\underline{A} = 0$. Hence Y is a continuous semimartingale with canonical decomposition

$$Y. = \frac{X.(Y) - X.(-Y)}{2} = Y_0 - A. + \overline{M}.,$$

where $-A := \frac{-\overline{A} + \underline{A}}{2}$. It thus remains to show that A satisfies the structure condition $\mathcal{Q}(\Lambda, C)$. From (4.1) we obtain

$$2dD + d\langle \overline{M} \rangle = \underbrace{\left(-d\overline{A} + \frac{1}{2}d\langle \overline{M} \rangle\right)}_{\gg 0} + \underbrace{\left(-d\underline{A} + \frac{1}{2}d\langle \overline{M} \rangle\right)}_{\gg 0}.$$

By Radon-Nikodým theorem there exists a predictable process α valued in $[0, 1]$ such that

$$\begin{aligned} d\left(-\overline{A} + \frac{1}{2}\langle \overline{M} \rangle\right) &= \alpha d(2D + \langle \overline{M} \rangle), \\ d\left(-\underline{A} + \frac{1}{2}\langle \overline{M} \rangle\right) &= (1 - \alpha) d(2D + \langle \overline{M} \rangle). \end{aligned}$$

This gives

$$-dA = (2\alpha - 1)d\left(D + \frac{1}{2}\langle \overline{M} \rangle\right).$$

Hence

$$d|A| \ll dD + \frac{1}{2}d\langle \overline{M} \rangle.$$

(iii). It remains to prove the rest statement. Suppose Y is $\mathcal{Q}(\Lambda, C)$ semimartingale, then $U(e^Y)$ is a continuous semimartingale. Itô's formula applied to $U(e^Y)$ and $X.(Y) = Y. + D.$ imply

$$\begin{aligned} dU.(e^Y) &= de^{Y.} + e^{Y.}dD. \\ &= de^{X.(Y) - D.} + e^{Y.}dD. \\ &= e^{-D.}de^{X.(Y)} + e^{X.(Y)}de^{-D.} + e^{Y.}dD. \\ &= e^{-D}de^{X.(Y)}. \end{aligned}$$

Hence, $U(e^Y)$ is a continuous local submartingale by (i). The same arguments also apply to $U(e^{-Y})$. For the converse direction, we show analogously to (ii) that Y is a continuous semimartingale by Mertens decomposition of $U(e^Y)$ and $U(e^{-Y})$. Therefore, $X(Y)$ and $X(-Y)$ are both continuous semimartingales. Again Itô's formula used to $e^{X(Y)}$ gives

$$de^{X.(Y)} = e^{D.}dU.(e^Y).$$

Hence $X(Y)$ is \mathcal{Q} submartingale. The same arguments also apply to $X(-Y)$. Finally by (ii) we conclude that Y is $\mathcal{Q}(\Lambda, C)$ semimartingale.

□

For later use, for any optional process Y , we define

$$\bar{X}^{\Lambda, C}(Y) := e^{C \cdot} |Y| + \int_0^\cdot e^{C_s} d\Lambda_s.$$

Sometimes only the terminal value of this process matters. Hence we use the same notation $X_T^{\Lambda, C}$ to define, for any \mathcal{F}_T -measurable random variable Ξ ,

$$\bar{X}_T^{\Lambda, C}(|\Xi|) := e^{C_T} |\Xi| + \int_0^T e^{C_s} d\Lambda_s.$$

Proposition 4.2 *If Y is a $\mathcal{Q}(\Lambda, C)$ semimartingale, then $\bar{X}(Y)$ is a continuous \mathcal{Q} submartingale.*

Proof. By Itô's formula,

$$\begin{aligned} d\bar{X}(Y) &= e^C \left(|Y| dC + d\Lambda - \operatorname{sgn}(Y) dA + \operatorname{sgn}(Y) dM + dL(Y) \right) \\ &= e^C \left(dD - \operatorname{sgn}(Y) dA + dL(Y) \right) + e^C \operatorname{sgn}(Y) dM, \end{aligned}$$

where $L(Y)$ is the local time of Y at 0. By the structure condition $\mathcal{Q}(\Lambda, C)$, $\bar{X}(Y)$ is a continuous \mathcal{Q} submartingale.

□

Analogously we deduce that

$$e^{C_{u \cdot}} |Y| + \int_u^\cdot e^{C_{u,s}} d\Lambda_s = e^{-C_u} \cdot (\bar{X}_\cdot(Y) - \int_0^u e^{C_s} d\Lambda_s) \quad (4.2)$$

is a \mathcal{Q} submartingale on $[u, T]$ starting from $|Y_u|$ if we view $u \in [0, T]$ as the initial time.

4.3 Stability of Quadratic Semimartingales

Let us turn to the main study of this chapter: stability of quadratic semimartingales. To this end, we give some estimates used later to prove the convergence of quadratic semimartingales, their martingale parts and finite variation parts in suitable spaces.

Observe that the nonadapted continuous process $\phi^{\Lambda, C}(|Y_T|)$ defined by

$$\phi. := e^{C. \cdot T} |Y_T| + \int. e^{C. \cdot s} d\Lambda_s$$

is a positive decreasing solution to the ODE

$$d\phi_t = -(d\Lambda_t + |\phi_t| dC_t), \quad \phi_0 = \bar{X}_T(|Y_T|), \quad \phi_T = |Y_T|.$$

By differentiating $e^{\phi.}$ we obtain

$$e^{\phi_0} = e^{\phi.} + \underbrace{\int_0. e^{\phi_s} d\Lambda_s + \int_0. e^{\phi_s} |\phi_s| dC_s}_{:= A^{\phi}}. \quad (4.3)$$

Let us make the following standing assumption for the estimations.

Assumption. For a $\mathcal{Q}(\Lambda, C)$ semimartingale Y , set $\exp(\bar{X}_T^{\Lambda, C}(|Y_T|)) \in \mathbb{L}^1$ and define $\Phi^{\Lambda, C}(|Y_T|) := \mathbb{E}[\exp(\phi^{\Lambda, C}(|Y_T|)) | \mathcal{F}]$.

Theorem 4.3 Set $(\phi, \Phi) := (\phi^{\Lambda, C}(|Y_T|), \Phi^{\Lambda, C}(|Y_T|))$.

- (i) Φ is a continuous positive supermartingale of class \mathcal{D} with canonical decomposition $\Phi. = -A^{\Phi} + M^{\Phi}$, where M^{Φ} is a continuous martingale and $A^{\Phi} = \int_0. \Phi_s d\Lambda_s + \int_0. \mathbb{E}[\exp(\phi_s) |\phi_s| | \mathcal{F}_s] dC_s$.
- (ii) $U.(\Phi) = \Phi. + \int_0. \Phi_s d\Lambda_s + \int_0. \Phi_s \ln(\Phi_s) dC_s$ is a continuous positive supermartingale of class \mathcal{D} with canonical decomposition $U.(\Phi) = -A^U + M^U$, where $M^U = M^{\Phi}$ and $A^U = \int_0. (\mathbb{E}[\exp(\phi_s) |\phi_s| | \mathcal{F}_s] - \Phi_s \ln(\Phi_s)) dC_s$.
- (iii) If in addition $\exp(|Y.|) \leq \Phi.$, then the processes $U(e^Y)$ and $U(e^{-Y})$ are continuous submartingales of class \mathcal{D} dominated by $U(\Phi)$.

Proof. (i). For any $\tau, \sigma \in \mathcal{T}$, $\tau \leq \sigma$, $\phi.$ being decreasing yields

$$\mathbb{E}[\exp(\phi_0) | \mathcal{F}_\tau] \geq \mathbb{E}[\exp(\phi_\tau) | \mathcal{F}_\tau] = \Phi_\tau \geq \mathbb{E}[\mathbb{E}[\exp(\phi_\sigma) | \mathcal{F}_\sigma] | \mathcal{F}_\tau] = \mathbb{E}[\Phi_\sigma | \mathcal{F}_\tau].$$

Hence Φ is a supermartingale of class \mathcal{D} which is also the optional projection of $\exp(\phi.)$. Moreover, since Λ and C are continuous, nondecreasing and adapted, the dual predictable projection of A^{ϕ} in (4.3) is A^{Φ} with $\mathbb{E}[A_{t,T}^{\phi} - A_{t,T}^{\Phi} | \mathcal{F}_t] = 0$. Hence $\widetilde{M} := \mathbb{E}[A_T^{\Phi} - A_T^{\phi} | \mathcal{F}_t] = \mathbb{E}[A_t^{\phi} - A_t^{\Phi} | \mathcal{F}_t]$ is a martingale. Hence (4.3) gives

$$\mathbb{E}[\exp(\phi_0) | \mathcal{F}_t] = \Phi_t + \mathbb{E}[A_t^{\phi} | \mathcal{F}_t] = \Phi_t + \widetilde{M}_t + A_t^{\Phi}.$$

Then (i) is immediate by setting $M^{\Phi} := \mathbb{E}[\exp(\phi_0) | \mathcal{F}] - \widetilde{M}$. The continuity simply comes from the continuity of A^{Φ} and M^{Φ} .

(ii). By (i), $U(\Phi)$ is a continuous positive semimartingale with canonical decomposition $U(\Phi) = -A^U + M^U$, where $M^U = M^\Phi$ and $A^U = \int_0^\cdot (\mathbb{E}[e^{\phi_s} | \mathcal{F}_s] - \Phi_s \ln(\Phi_s)) dC_s$. By Jensen's inequality, A^U is nondecreasing, hence $U(\Phi)$ is a supermartingale. The class \mathcal{D} property comes from the fact that $U(\Phi)$ is dominated by M^Φ .

(iii). This directly comes from $e^{|Y|} \leq \Phi$, (ii) and characterizations of $\mathcal{Q}(\Lambda, C)$ semimartingales (Theorem 4.1). □

Remark. A sufficient and necessary condition to verify $e^{|Y|} \leq \Phi$ in Theorem 4.3(iii) is that $\exp(\bar{X}(Y))$ is of class \mathcal{D} . Indeed, if $\exp(\bar{X}(Y))$ is of class \mathcal{D} , $\exp(|Y|) \leq \Phi$ is immediate from the \mathcal{Q} submartingale property in (4.2). For the converse direction we assume $\exp(|Y|) \leq \Phi = \mathbb{E}[\exp(\phi_0) | \mathcal{F}]$. Taking power e^C on both sides, using Jensen's inequality to the right-hand side and finally multiplying both sides by $\exp(\int_0^\cdot e^{C_s} d\Lambda_s)$ yields $\exp(\bar{X}(Y)) \leq \mathbb{E}[\exp(\phi_0) | \mathcal{F}]$. Hence $\exp(\bar{X}(Y))$ is of class \mathcal{D} .

In applications $e^{|Y|} \leq \Phi$ is natural and often satisfied. For example, in BSDE framework, $e^{|Y|} \leq \Phi$ can be seen as an estimate for the solution process Y .

If Ξ is a \mathcal{F}_T -measurable random variable such that $\exp(\bar{X}_T^{\Lambda, C}(|\Xi|)) \in \mathbb{L}^1$, then Theorem 4.3 still holds. Hence it is a common property of $\mathcal{Q}(\Lambda, C)$ semimartingales whose terminal values are bounded by $|\Xi|$. This fact will be used to prove the stability result.

Given stronger integrability condition on $\bar{X}_T^{\Lambda, C}(|Y_T|)$ we can prove a maximal inequality for $\mathcal{Q}(\Lambda, C)$ semimartingales. The proof essentially relies on Proposition 4.2 which states that $\bar{X}^{\Lambda, C}(Y)$ is a \mathcal{Q} submartingale dominating Y . To this end we define $\psi_p = x^p$ for $p \neq 1$ and $\psi_1 = x \ln x - x + 1$ for $x \in \mathbb{R}^+$.

Lemma 4.4 (Maximal Inequality) *Let $p \geq 1$. If Y is a $\mathcal{Q}(\Lambda, C)$ semimartingale such that $\psi_p(\bar{X}_T(|Y_T|)) \in \mathbb{L}^1$, then*

(i) $\mathbb{E}[\exp(pY^*)]^{\frac{1}{p}}$ is dominated by some increasing function of

$$\mathbb{E}\left[\psi_p\left(\exp\left(\bar{X}_T(|Y_T|)\right)\right)\right].$$

(ii) For any $0 < q < 1$, $\mathbb{E}[\exp(qY^*)]$ is dominated by some increasing function of

$$\psi_q\left(\mathbb{E}\left[\exp\left(\bar{X}_T(|Y_T|)\right)\right]\right).$$

Proof. The proof is based on various maximal inequalities and omitted here since it is not relevant to our study of the stability result. For details the reader shall refer to Proposition 3.4 and Proposition 3.5, Barrieu and El Karoui [4]. □

Given the above estimates we are ready to introduce a stable family of quadratic semimartingales. The stability result consists in proving convergence of quadratic semimartingales, their finite variation parts and martingale parts.

$\mathcal{S}_Q(|\Xi|, \Lambda, C)$ Class. Let Ξ be a \mathcal{F}_T -measurable random variable with $\exp(\bar{X}_T(|\Xi|)) \in \mathbb{L}^1$. Define $\mathcal{S}_Q(|\Xi|, \Lambda, C)$ to be the set of $\mathcal{Q}(\Lambda, C)$ semimartingales Y with $|Y_T| \leq |\Xi|$ such that $\exp(|Y|) \leq \Phi(|Y_T|)$. By the remark after Theorem 4.3, this inequality is equivalent to $\exp(\bar{X} \cdot (Y))$ being of class \mathcal{D} . Define $\mathbf{P} := \{p \in \mathbb{R}^+ : \mathbb{E}[\exp(p\bar{X}_T(|\Xi|))] < +\infty\}$ and $p^* := \sup \mathbf{P}$. It is obvious that $1 \in \mathbf{P}$ and $p^* \geq 1$.

To prepare for the stability result, we give some estimates for the finite variation parts and the martingale parts in the next two lemmas.

Lemma 4.5 (Estimate) *Let $Y \in \mathcal{S}_Q(|\Xi|, \Lambda, C)$ with canonical decomposition $Y = Y_0 - A + M$. Set $(\bar{X}_T, \Phi) := (\bar{X}_T^{\Lambda, C}(|\Xi|), \Phi^{\Lambda, C}(|\Xi|))$.*

(i) *For any $\tau \in \mathcal{T}$,*

$$\frac{1}{2}\mathbb{E}[\langle M \rangle_{\tau, T} | \mathcal{F}_\tau] \leq \Phi_\tau \mathbb{I}_{\{\tau < T\}} \leq \mathbb{E}[\exp(\bar{X}_T) \mathbb{I}_{\{\tau < T\}} | \mathcal{F}_\tau].$$

In particular,

$$\mathbb{E}[\langle M \rangle_T] \leq 2\mathbb{E}[\exp(\bar{X}_T)].$$

(ii) *If Φ is bounded, then M is a BMO martingale.*

(iii) *For any $p \in \mathbf{P} \cap [1, +\infty)$, $M \in \mathcal{M}^{2p}$ with*

$$\mathbb{E}[\langle M \rangle_T^p] \leq (2p)^p \mathbb{E}[\exp(p\bar{X}_T)].$$

Proof. (i). Define $u(x) := e^x - x - 1$. Hence, $u(x) \geq 0$, $u'(x) \geq 0$ and $u''(x) \geq 1$ for $x \geq 0$; $u \in \mathcal{C}^2(\mathbb{R})$ and $u'' - u' = 1$. For any $\tau, \sigma \in \mathcal{T}$, Itô's formula and the structure condition $\mathcal{Q}(\Lambda, C)$ yield

$$\frac{1}{2}\langle M \rangle_{\tau \wedge \sigma, \sigma} \leq u(|Y_\sigma|) - u(|Y_{\tau \wedge \sigma}|) + \int_{\tau \wedge \sigma}^{\sigma} u'(|Y_s|) dD_s - \int_{\tau \wedge \sigma}^{\sigma} u'(|Y_s|) dM_s.$$

$\exp(\bar{X}^{\Lambda, C} \cdot (Y))$ being of class \mathcal{D} implies that $\exp(|Y|)$ is of class \mathcal{D} . To eliminate local martingale we replace σ by its localizing sequence. By Fatou's lemma and class \mathcal{D} property of $\exp(|Y|)$,

$$\frac{1}{2}\mathbb{E}[\langle M \rangle_{\tau, T} | \mathcal{F}_\tau] \leq \mathbb{E}\left[u(|Y_T|) - u(|Y_\tau|) + \int_{\tau}^T u'(|Y_s|) dD_s \middle| \mathcal{F}_\tau\right].$$

By $u'(|Y_s|) \leq \exp(|Y_s|) \leq \Phi_s^{\Lambda, C}(|Y_T|) \leq \Phi_s$,

$$\int_{\tau}^T u'(|Y_s|) dD_s \leq \int_{\tau}^T \Phi_s d\Lambda_s + \Phi_s \ln(\Phi_s) dC_s.$$

Since $U(\Phi) = \Phi + \int_0^\cdot \Phi_s d\Lambda_s + \int_0^\cdot \Phi_s \ln(\Phi_s) dC_s$ is a supermartingale by Theorem 4.3(ii),

$$\begin{aligned}
\frac{1}{2}\mathbb{E}[\langle M \rangle_{\tau,T} | \mathcal{F}_\tau] &\leq \mathbb{E}\left[u(|Y_T|) - u(|Y_\tau|) + \int_\tau^T \Phi_s d\Lambda_s + \Phi_s \ln(\Phi_s) dC_s \middle| \mathcal{F}_\tau\right] \\
&\leq \mathbb{E}[u(|Y_T|) - \Phi_T - u(|Y_\tau|) + \Phi_\tau | \mathcal{F}_\tau] \\
&\leq \Phi_\tau \mathbb{I}_{\{\tau < T\}} \\
&\leq \mathbb{E}[\exp(\bar{X}_T) \mathbb{I}_{\{\tau < T\}} | \mathcal{F}_\tau],
\end{aligned} \tag{4.4}$$

where the third inequality is due to $u(|Y_T|) \leq \Phi_T$ and $u(|Y_\tau|) \geq 0$.

(ii). This is immediate from (i).

(iii). This is immediate from Garsia-Neveu lemma (see Chapter VI, Dellacherie and Meyer [13]) applied to (i).

□

Lemma 4.6 (Estimate) *Let $Y \in \mathcal{S}_Q(|\Xi|, \Lambda, C)$ with canonical decomposition $Y = Y_0 - A + M$. Set $(\bar{X}_T, \Phi) := (\bar{X}_T^{\Lambda, C}(|\Xi|), \Phi^{\Lambda, C}(|\Xi|))$.*

(i) *For any $\tau \in \mathcal{T}$,*

$$\mathbb{E}[|A|_{\tau,T} | \mathcal{F}_\tau] \leq 2\mathbb{E}[\exp(\bar{X}_T) \mathbb{I}_{\{\tau < T\}} | \mathcal{F}_\tau].$$

In particular,

$$\mathbb{E}[|A|_T] \leq 2\mathbb{E}[\exp(\bar{X}_T)].$$

(ii) *If Φ is bounded by c_Φ , then for any $\tau \in \mathcal{T}$,*

$$\mathbb{E}[|A|_{\tau,T} | \mathcal{F}_\tau] \leq 2c_\Phi.$$

(iii) *For any $p \in \mathbf{P} \cap [1, +\infty)$, the total variation of A satisfies*

$$\mathbb{E}[|A|_T^p] \leq (2p)^p \mathbb{E}[\exp(p\bar{X}_T)].$$

Proof. (i). By the structure condition $\mathcal{Q}(\Lambda, C)$, $\exp(|Y|) \leq \Phi$, supermartingale property of $U(\Phi)$ and (4.4), we have

$$\begin{aligned}
\mathbb{E}[|A|_{\tau,T} | \mathcal{F}_\tau] &\leq \mathbb{E}\left[\Lambda_{\tau,T} + \int_\tau^T |Y_s| dC_s \middle| \mathcal{F}_\tau\right] + \frac{1}{2}\mathbb{E}[\langle M \rangle_{\tau,T} | \mathcal{F}_\tau] \\
&\leq \mathbb{E}\left[\int_\tau^T e^{|Y_s|} (d\Lambda_s + |Y_s| dC_s) \middle| \mathcal{F}_\tau\right] + \Phi_\tau \mathbb{I}_{\{\tau < T\}} \\
&\leq \mathbb{E}\left[\int_\tau^T \Phi_s (d\Lambda_s + \ln(\Phi_s) dC_s) \middle| \mathcal{F}_\tau\right] + \Phi_\tau \mathbb{I}_{\{\tau < T\}} \\
&\leq \mathbb{E}[\Phi_\tau - \Phi_T | \mathcal{F}_\tau] + \Phi_\tau \mathbb{I}_{\{\tau < T\}} \\
&\leq 2\mathbb{E}[\exp(\bar{X}_T) \mathbb{I}_{\{\tau < T\}} | \mathcal{F}_\tau].
\end{aligned}$$

- (ii). This is immediate from (i).
- (iii). This is immediate from Garsia-Neveu lemma.

□

With the estimates for the finite variation parts and the martingale parts, we are ready to prove the stability result. We start by showing that $\mathcal{S}_{\mathcal{Q}}(|\Xi|, \Lambda, C)$ is stable by \mathbb{P} -a.s. convergence.

Stability of $\mathcal{S}_{\mathcal{Q}}(|\Xi|, \Lambda, C)$. Let $\{Y^n\}_{n \in \mathbb{N}^+} \subset \mathcal{S}(|\Xi|, \Lambda, C)$ and assume \mathbb{P} -a.s. Y^n converges to Y on $[0, T]$ as n goes to $+\infty$. By Theorem 4.3(iii), the continuous submartingales $U(Y^n)$ and $U(-Y^n)$ are dominated by the positive supermartingale $U(\Phi(|\Xi|))$. Hence, by dominated convergence, we can pass the submartingale property to $U(Y)$ and $U(-Y)$. Clearly $U(Y)$ and $U(-Y)$ are optional since they are limit of continuous submartingales. Then by characterizations of $\mathcal{Q}(\Lambda, C)$ semimartingales (Theorem 4.1), Y is a $\mathcal{Q}(\Lambda, C)$ semimartingale. Moreover, taking the limit also yields $|Y_T| \leq |\Xi|$ and $\exp(|Y|) \leq \Phi(|Y_T|)$. Hence $Y \in \mathcal{S}_{\mathcal{Q}}(|\Xi|, \Lambda, C)$. In addition, if the convergence is monotone, Dini's theorem implies that the convergence is \mathbb{P} -a.s. uniform on $[0, T]$.

Given the estimates for Y^n, A^n and M^n , the following theorem states that A^n and M^n also converge in suitable spaces.

Theorem 4.7 (Stability) *Let $\{Y^n\}_{n \in \mathbb{N}^+} \subset \mathcal{S}_{\mathcal{Q}}(|\Xi|, \Lambda, C)$ with canonical decomposition $Y^n = Y_0 - A^n + M^n$. If Y^n is Cauchy for \mathbb{P} -a.s. uniform convergence on $[0, T]$, then the limit process Y belongs to $\mathcal{S}_{\mathcal{Q}}(|\Xi|, \Lambda, C)$. Denote its canonical decomposition by $Y = Y_0 - A + M$.*

- (i) For any $1 \leq p < 2$, M^n converges to M in \mathcal{M}^p .
- (ii) If $p^* > 1$, then for any $1 \leq p < p^*$, M^n converges to M in \mathcal{M}^{2p} .
- (iii) In both cases, A^n converges at least in \mathcal{S}^1 to A .

Proof. (i). Set $1 \leq p < 2$. Define $\delta Y^{m,n} := Y^m - Y^n$ and $\delta A^{m,n}, \delta M^{m,n}$, etc. analogously. For each $k \in \mathbb{N}^+$, set $\tau_k := \inf \{t \geq 0 : \langle \delta M^{m,n} \rangle_t \geq k\} \wedge T$. By Itô's formula,

$$|\delta Y_0^{m,n}|^2 + \langle \delta M^{m,n} \rangle_{\tau_k} \leq |\delta Y_{\tau_k}^{m,n}|^2 + 2 \int_0^{\tau_k} |\delta Y_s^{m,n}| d|\delta A^{m,n}|_s + 2 \left| \int_0^{\tau_k} \delta Y_s^{m,n} dM_s^{m,n} \right|.$$

By Davis-Burkholder-Gundy inequality and Cauchy-Schwartz inequality,

$$\begin{aligned} \mathbb{E}[\langle \delta M^{m,n} \rangle_{\tau_k}^{\frac{p}{2}}] &\leq \mathbb{E}[(\langle \delta Y^{m,n} \rangle^*)^p] + 2^{\frac{p}{2}} \mathbb{E}[(\langle \delta Y^{m,n} \rangle^*)^{\frac{p}{2-p}}]^{\frac{2-p}{2}} \mathbb{E}[|\delta A^{m,n}|_T]^{\frac{p}{2}} \\ &\quad + \left(c(p) \mathbb{E}[\langle \delta Y^{m,n} \rangle^*]^{\frac{1}{2}} \right) \mathbb{E}[\langle \delta M^{m,n} \rangle_{\tau_k}^{\frac{p}{2}}]^{\frac{1}{2}} \\ &< +\infty. \end{aligned}$$

Here $c(p)$ denotes the constant from Davis-Burkholder-Gundy inequality which only depends on p . Using $ab \leq \frac{a^2+b^2}{2}$, we obtain by transferring $\mathbb{E}[\langle \delta M^{m,n} \rangle_{\tau_k}^{\frac{p}{2}}]$ to the left-hand side and Fatou's lemma that

$$\mathbb{E}[\langle \delta M^{m,n} \rangle_T^{\frac{p}{2}}] \leq (2 + c(p)^2) \mathbb{E}[(\delta Y^{m,n})^*]^p + 2^{\frac{p+2}{2}} \mathbb{E}[(\delta Y^{m,n})^*]^{\frac{p}{2-p}} \mathbb{E}[|\delta A^{m,n}|_T]^{\frac{p}{2}}.$$

By Lemma 4.6(i), $\mathbb{E}[|\delta A^{m,n}|_T]$ is uniformly bounded. Moreover, Lemma 4.4(ii) implies by de la Vallée-Poussin criterion that for any $r > 0$, $((\delta Y^{m,n})^*)^r$ is uniformly integrable. Hence Vitali convergence implies that M^n is Cauchy in \mathcal{M}^p . The \mathcal{M}^p -limit of M^n coincides with M since the canonical decomposition of Y is unique.

(ii). For any $\tau, \sigma \in \mathcal{T}$, Itô's formula yields

$$\begin{aligned} |\delta Y_{\tau \wedge \sigma}^{m,n}|^2 + \langle \delta M^{m,n} \rangle_{\tau \wedge \sigma, \sigma} &= |\delta Y_{\sigma}^{m,n}|^2 + 2 \int_{\tau \wedge \sigma}^{\sigma} \delta Y_s^{m,n} d(\delta A_s^{m,n} - \delta M_s^{m,n}) \\ &\leq ((\delta Y^{m,n})^*)^2 + 2 \int_0^T |\delta Y_s^{m,n}| d|\delta A^{m,n}|_s - 2 \int_{\tau \wedge \sigma}^{\sigma} \delta Y_s^{m,n} d\delta M_s^{m,n}. \end{aligned}$$

To eliminate the local martingale we replace σ by its localizing sequence. Then Fatou's lemma yields

$$\mathbb{E}[\langle \delta M^{m,n} \rangle_{\tau, T} | \mathcal{F}_{\tau}] \leq \mathbb{E}\left[\left((\delta Y^{m,n})^*\right)^2 + 2(\delta Y^{m,n})^* |\delta A^{m,n}|_T\right] \mathbb{I}_{\{\tau < T\}} \Big| \mathcal{F}_{\tau}.$$

For any p such that $1 \leq p < p^*$, we can find $\epsilon > 0$ such that $1 \leq p < p + \epsilon < p^*$. By Garsia-Neveu lemma and Hölder inequality, the above estimate gives

$$\begin{aligned} \mathbb{E}[\langle \delta M^{m,n} \rangle_T^p] &\leq p^p \mathbb{E}\left[\left((\delta Y^{m,n})^*\right)^2 + (\delta Y^{m,n})^* |\delta A^{m,n}|_T\right]^p \\ &\leq p^p 2^{p-1} \left(\mathbb{E}[(\delta Y^{m,n})^*]^{2p} + \mathbb{E}[(\delta Y^{m,n})^*]^p \mathbb{E}[|\delta A^{m,n}|_T^p]\right) \\ &\leq p^p 2^{p-1} \left(\mathbb{E}[(\delta Y^{m,n})^*]^{2p} + \mathbb{E}[(\delta Y^{m,n})^*]^{\frac{p(p+\epsilon)}{\epsilon}} \mathbb{E}[|\delta A^{m,n}|_T^{p+\epsilon}]^{\frac{p}{p+\epsilon}}\right). \end{aligned}$$

Since $p + \epsilon < p^*$, $\mathbb{E}[|\delta A^{m,n}|_T^{p+\epsilon}]$ is uniformly bounded due to Lemma 4.6(iii). Hence Vitali convergence gives the result.

(iii). This is immediate from (i)(ii). □

Remark. The case where $p = 1$ in Theorem 4.7(i) is also a consequence of Barlow and Protter [2] which proves convergence of the martingale parts in \mathcal{M}^1 for semimartingales.

4.4 Applications to Quadratic BSDEs

Based on the stability result of quadratic semimartingales obtained in Section 4.3, we study the corresponding monotone stability result for quadratic BSDEs. Here we continue with the continuous semimartingale setting in Chapter 3.

Recall that the BSDE (f, g, ξ) is written as follows

$$Y = Y_0 - \underbrace{\int_0^\cdot ((f(s, Y_s, Z_s) dA_s + g_s d\langle N \rangle_s))}_{:= \widetilde{A}} + \underbrace{\int_0^\cdot (Z_s dM_s + dN_s)}_{:= \widetilde{M}}, \quad Y_T = \xi, \quad (4.5)$$

where $Y = Y_0 - \widetilde{A} + \widetilde{M}$ is the canonical decomposition. Without loss of generality we assume \mathbb{P} -a.s. $|g| \leq \frac{1}{2}$. Let α be an \mathbb{R} -valued Prog-measurable process and $\beta \geq 0$. If the structure condition

$$|f(t, y, z)| \leq \alpha_t + \beta|y| + \frac{1}{2}|\lambda_t z|^2, \quad (4.6)$$

holds, then

$$\begin{aligned} |f(t, Y_t, Z_t)| dA_t &\ll \left(\alpha_t + \beta|Y_t| + \frac{1}{2}|\lambda_t Z_t|^2 \right) dA_t \\ &= \Lambda_t + |Y_t| dC_t + \frac{1}{2} d\langle Z \cdot M \rangle_t, \end{aligned}$$

where $\Lambda := \alpha \cdot A$, $C := \beta A$. Hence

$$d|\widetilde{A}| \ll \Lambda + |Y| dC + \frac{1}{2} d\langle \widetilde{M} \rangle.$$

Thus if (Y, Z, N) is a solution to (4.5) which satisfies (4.6), then Y is a $\mathcal{Q}(\Lambda, C)$ semimartingale. This motivates us to convert the machinery of quadratic semimartingales into a monotone stability result for quadratic BSDEs.

Proposition 4.8 (Monotone Stability) *Let (Y^n, Z^n, N^n) be solutions to (f^n, g^n, ξ^n) for each $n \in \mathbb{N}^+$, respectively, and Y^n be a monotone sequence in $\mathcal{S}_{\mathcal{Q}}(|\Xi|, \Lambda, C)$ which converges \mathbb{P} -a.s. to Y . Denote their canonical decomposition by $Y^n = Y_0^n - \widetilde{A}^n + \widetilde{M}^n$, where $\widetilde{M}^n = Z^n \cdot M + N^n$.*

- (i) *Then $Y \in \mathcal{S}_{\mathcal{Q}}(|\Xi|, \Lambda, C)$ and the convergence is \mathbb{P} -a.s. uniform on $[0, T]$. Denote its canonical decomposition by $Y := Y_0 - \widetilde{A} + \widetilde{M}$. Then \widetilde{M}^n converges in \mathcal{M}^p to \widetilde{M} for any $1 \leq p < 2$. Moreover, \widetilde{M} admits a decomposition $\widetilde{M} = Z \cdot M + N$.*
- (ii) *If (f^n, g^n, ξ^n) satisfies (4.6) and \mathbb{P} -a.s. for any $t \in [0, T]$, $y^n \rightarrow y$, $z^n \rightarrow z$, $f^n(t, y_n, z_n) \rightarrow f(t, y, z)$ and $g_t^n \rightarrow g_t$, then (Y, Z, N) solves $(f, g, \lim_n \xi^n)$.*

Proof. (i). This is immediate from the stability result of $\mathcal{S}_{\mathcal{Q}}(\Lambda, C, |\Xi|)$ and Theorem 4.7(i).

(ii). Given the convergence of Y^n and \widetilde{M}^n , it remains to prove \widetilde{A}^n converges *u.c.p* to \widetilde{A} , which consists of proving

$$\int_0^\cdot g_s^n d\langle N^n \rangle_s \longrightarrow \int_0^\cdot g_s d\langle N \rangle_s, \quad \int_0^\cdot f^n(s, Y_s^n, Z_s^n) dA_s \longrightarrow \int_0^\cdot f(s, Y_s, Z_s) dA_s \quad (4.7)$$

u.c.p. as n goes to $+\infty$. To prove the first convergence result, Kunita-Watanabe inequality and Cauchy-Schwartz inequality yield

$$\begin{aligned} & \mathbb{E} \left[\left(\left| \int_0^\cdot (g_s^n d\langle N^n \rangle_s - g_s d\langle N \rangle_s) \right|^* \right)^{\frac{1}{2}} \right] \\ & \leq \mathbb{E} \left[\left| \int_0^T |g_s^n| |d\langle N^n \rangle - \langle N \rangle|_s \right|^{\frac{1}{2}} \right] + \mathbb{E} \left[\left(\int_0^T |g_s^n - g_s| d\langle N \rangle_s \right)^{\frac{1}{2}} \right] \\ & \leq \frac{1}{2} \mathbb{E} [\langle N^n - N \rangle_T^{\frac{1}{2}}]^{\frac{1}{2}} \mathbb{E} [\langle N^n + N \rangle_T^{\frac{1}{2}}]^{\frac{1}{2}} + \mathbb{E} \left[\left(\int_0^T |g_s^n - g_s| d\langle N \rangle_s \right)^{\frac{1}{2}} \right]. \end{aligned}$$

By Lemma 4.5(i), $\mathbb{E} [\langle N^n + N \rangle_T^{\frac{1}{2}}]^{\frac{1}{2}}$ is uniformly bounded. By Theorem 4.7(i), N^n converges to N in \mathcal{M}^1 . For the second term, we use dominated convergence.

To prove the second convergence result in (4.7), we use a localization procedure. Note that $\Psi_\cdot := \mathbb{E} [\exp(\phi_0(\Xi)) | \mathcal{F}_\cdot]$ is a continuous martingale due to the continuity of the filtration. For each $n \in \mathbb{N}^+$, define $\sigma_k := \inf \{t \geq 0 : \Psi_t \geq k\}$. The definition of $\mathcal{S}_{\mathcal{Q}}(|\Xi|, \Lambda, C)$ then implies that $\exp(Y_{\cdot \wedge \sigma_k}^n) \leq \Phi_{\cdot \wedge \sigma_k} \leq \Psi_{\cdot \wedge \sigma_k} \leq k$. Secondly, by Lemma 4.5(i), \widetilde{M}^n and hence $Z^n \cdot M$ are uniformly bounded in \mathcal{M}^2 . Moreover, since \widetilde{M}^n converges to \widetilde{M} in \mathcal{M}^1 by Theorem 4.7(i), we can assume Z^n converges to Z $d\langle M \rangle \otimes d\mathbb{P}$ -a.e. and

$$\mathbb{E} \left[\int_0^T \sup_n |Z_s^n|^2 d\langle M \rangle_s \right] < +\infty,$$

by substracting a subsequence; see Lemma 2.5, Kobylanski [22]. By dominated convergence, $\int_0^{\sigma_k} |f^n(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s)| dA_s$ converges to 0 as n goes to $+\infty$. Hence the second convergence result in (4.7) is immediate.

□

The stability result in this section gives a forward point of view to answer the question of convergence. In contrast to Kobylanski [22], it allows unboundedness and proves the stability of $\mathcal{S}_{\mathcal{Q}}(\Lambda, C, |\Xi|)$ which is later used to show $\mathcal{M}^p(p \geq 1)$ convergence of the martingale parts. Nevertheless the structure condition $\mathcal{Q}(\Lambda, C)$ requires a linear growth in y which is crucial to the estimate for the finite variation parts; see Lemma 4.6. Hence, given general growth conditions (see, e.g., Briand and Hu [9] or Section 3.3, 3.4) where

the estimate for A is not available, it is more difficult to derive the stability result with the help of quadratic semimartingales.

To end this section we give an existence example where boundedness as required by classic existence results is no longer needed.

Existence: an Example. Let the BSDE $(f, 0, \xi)$ satisfy (4.6) and $\exp(\bar{X}_T^{\Lambda, C}(|\xi|)) \in \mathbb{L}^1$. We show that there exists a solution to $(f, 0, \xi)$ by Proposition 4.8 (monotone stability). For each $n, k \in \mathbb{N}^+$, define

$$\begin{aligned} f^{n,k}(t, y, z) : &= \inf_{y', z'} \{ f^+(t, y', z') + n|y - y'| + n|\lambda_t(z - z')| \} \\ &\quad - \inf_{y', z'} \{ (f^-(t, y', z') + k|y - y'| + k|\lambda_t(z - z')|) \}. \end{aligned}$$

By Lepeltier and San Martin [23], $f^{n,k}$ satisfies (4.6) and is Lipschitz-continuous in (y, z) . Moreover, $\exp(\bar{X}_T^{\Lambda, C}(|\xi|)) \in \mathbb{L}^1$ implies $\xi, |\alpha|_T \in \mathbb{L}^2$. Hence, by El Karoui and Huang [15], there exists a unique solution $(Y^{n,k}, Z^{n,k}, N^{n,k})$ to $(f^{n,k}, 0, \xi)$. To prove $Y^{n,k} \in \mathcal{S}_{\mathcal{Q}}(\Lambda, C, |\xi|)$, it remains to show $\exp(|Y^{n,k}|) \leq \Phi(|\xi^{n,k}|)$. First of all we assume $\bar{X}_T^{\Lambda, C}(|\xi|)$ is bounded. Then $Y^{n,k}$ is bounded and this inequality holds due to the class \mathcal{D} property of $\bar{X}^{\Lambda, C}(Y^{n,k})$. Note that the above inequality is stable when taking the limit in $\bar{X}_T^{\Lambda, C}(|\xi|)$, hence the inequality also holds for $Y^{n,k}$ with $\exp(\bar{X}_T^{\Lambda, C}(|\xi|)) \in \mathbb{L}^1$.

Given $(Y^{n,k})_{n,k \in \mathbb{N}^+} \subset \mathcal{S}_{\mathcal{Q}}(\Lambda, C, |\xi|)$, we are ready to construct a solution by a double approximation procedure. By comparison theorem, $Y^{n,k}$ is decreasing in k and increasing in n . Now we fix n . $\exp(|Y^{n,k}|) \leq \Phi(|\xi^{n,k}|) \leq \Phi(|\xi|)$ implies that the limit of $Y^{n,k}$ as k goes to $+\infty$ exists. We then use Proposition 4.8 to deduce the existence of a solution to $(f^{n,\infty}, 0, \xi)$. We denote it by $(Y^{n,\infty}, Z^{n,\infty}, N^{n,\infty})$. Thanks to the convergence of $Y^{n,k}$ we can pass the comparison property to $Y^{n,\infty}$. By exactly the same arguments as above, we construct a solution to $(f, 0, \xi)$ which is the limit of $(Y^{n,\infty}, Z^{n,\infty}, N^{n,\infty})$ as n goes to $+\infty$ in the sense of Proposition 4.8(i).

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